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Gauging the twisted Poincaré symmetry as a noncommutative theory of gravitation

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Einstein's theory of general relativity was formulated as a gauge theory of Lorentz symmetry by Utiyama in 1956, while the Einstein-Cartan gravitational theory was formulated by Kibble in 1961 as the gauge theory of Poincaré transformations. In this framework, we propose a formulation of the gravitational theory on canonical noncommutative space-time by covariantly gauging the twisted Poincaré symmetry, in order to fulfil the requirement of covariance under the general coordinate transformations, an essential ingredient of the theory of general relativity. It appears that the twisted Poincaré symmetry cannot be gauged by generalizing the Abelian twist to a covariant non-Abelian twist, nor by introducing a more general covariant twist element. The advantages of such a formulation as well as the related problems are discussed and possible ways out are outlined.

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I. INTRODUCTION

It is generally expected that the smooth manifold structure of the classical space-time should break down at distances of the order of the Planck length

$$l_{\mathrm{P}} = \sqrt{\frac{\hbar G}{c^3}} \approx 1.6 \cdot 10^{-35}\ \mathrm{m}, \tag{1.1}$$

so that all physical phenomena become essentially *nonlocal*—as opposed to the locality of traditional geometrical theories of gravitation and quantum and gauge field theories of particle physics. It is hoped that an appropriate implementation of the nonlocality will eventually enable the formulation of a unified theory of the fundamental interactions of nature, which should be free from singularities, divergences, and any other kind of inconsistencies. The noncommutativity of space-time coordinates is one way to implement the nonlocality of Planck scale physics, which is well motivated.

Formally, the noncommutativity of coordinate operators $\hat{x}^\mu$, $\mu = 0, 1, 2, 3$ is achieved by imposing the commutation relations

$$[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}, \tag{1.2}$$

where in the canonical case $\theta^{\mu\nu}$ is an antisymmetric constant matrix of dimension length squared, and by letting the fields on noncommutative space-time be functions of the noncommutative coordinate operators. Through Weyl quantization the noncommutative algebra of operators generated by (1.2) can be represented on the algebra of ordinary functions on classical space-time by using the noncommutative Moyal star product. The more general case with $\theta^{\mu\nu}$ being an antisymmetric tensor field has also been considered.

Combining Einstein's theory of general relativity and quantum mechanical measurements obeying Heisenberg's uncertainty principle leads to operational noncommutativity of space-time coordinates [1,2]. This has led to the formulation of quantum field theory on noncommutative space-time.

String theory is one of the strongest motivations for considering noncommutative space-time geometries and noncommutative gravitation. It has been shown that when the end points of strings in a theory of open strings are constrained to move on D branes in a constant B-field background and the theory is taken in a certain low-energy limit, then the full dynamics of the theory is described by a gauge theory on a noncommutative space-time [3]. Thus, noncommutative gauge theory emerges as a low-energy limit of open string theory with a constant antisymmetric background field.

The formulation of local (gauge) symmetries on a noncommutative (nonlocal) space-time is a delicate issue. Most gauge groups cannot be defined on noncommutative space-time, because they do not close under the star product. The noncommutative unitary group $U_\star(n)$ can be defined, but with representations limited by the no-go theorem [4] (see also [5,6]). A noncommutative standard model based on the gauge groups $U_\star(n)$ has been constructed [7] (see also, for its extension to noncommutative minimal supersymetric standard model, [8]).

Another approach to the noncommutative gauge theories has been through the so-called Seiberg-Witten map [3], which originally related a noncommutative $U_\star(n)$ gauge theory to a commutative one, both obtained as low-energy effective limits in string theory. The philosophy behind the Seiberg-Witten map has been subsequently used to formulate noncommutative gauge theories with gauge fields valued in the enveloping algebra of $su(n)$ [9,10] and a corresponding noncommutative version of the standard model has been built [11].

A new interpretation of the relativistic invariance of the commutation relations (1.2) was proposed in [12]: by using the concept of twisted Poincaré algebra, the relativistic invariance can be generalized to the framework of Hopf algebras. If in the usual (commutative) case, relativistic invariance means invariance under the Poincaré transformation, then in the noncommutative case relativistic invariance means invariance under twisted Poincaré transformations [13]. The representation content of the twisted Poincaré algebra is identical with that of the usual Poincaré algebra and this legitimates the usage of the familiar representations of the Poincaré symmetry in the context of noncommutative field theories [12]. The noncommutative field theories, although they lack the Lorentz symmetry, are invariant under the twisted Poincaré algebra, deformed by the Abelian twist element

$$\mathcal{F} = e^{(i/2)\theta^{\mu\nu}P_\mu\otimes P_\nu}, \tag{1.3}$$

where $P_\mu = -i\partial_\mu$ are the generators of space-time translations. The twist induces, on the representations of the Poincaré algebra, the deformed multiplication

$$\mu(\phi\otimes\psi) = \phi\psi \to \mu_\star(\phi\otimes\psi) = \mu(\mathcal{F}^{-1}(\phi\otimes\psi)) = \phi\star\psi, \tag{1.4}$$

which is precisely the (Moyal) star product. The question about the action of the twisted Poincaré algebra on fields and of the actual meaning of the invariance under twisted Poincaré algebra has been raised in [14,15].

Recently, an attempt was made to twist also the gauge symmetry, by extending the global Poincaré algebra through a semidirect product with the gauge algebra, and by twisting the coproduct of the combined algebra by using the Abelian twist element (1.3). This approach was shown [16] to be in conflict with the very idea of gauge symmetry, since it implicitly assumed that when a field transforms according to a given representation, then its partial derivatives of any order also transform in the same representation of the gauge algebra, which is obviously not the case. A further attempt [17] to twist in a gauge-covariant manner the internal gauge transformations and at the same time keep the Moyal space-time structure defined by (1.2) turned out to be unsuccessful. It is intriguing that the *external* Poincaré symmetry and the *internal* gauge symmetry cannot be unified under a common twist. This situation is reminiscent of the Coleman-Mandula theorem [18], although not entirely, since this theorem concerns global symmetry and simple groups. However, one can envisage that supersymmetry [19], due to its intrinsic internal symmetry, may reverse the situation, and a noncommutative gauge theory may be constructed by means of a twist.

There is a good understanding of the noncommutative effects on matter and gauge fields defined on the flat noncommutative space-time. The next step is to incorporate gravity by considering curved noncommutative spacetimes. The main problem is that the noncommutativity parameter $\theta^{\mu\nu}$ is usually taken to be constant, which breaks the Lorentz invariance of the commutation relations (1.2), and implicitly of any noncommutative field theory. This has motivated a large amount of work to study noncommutative deformations of general relativity (see, e.g., [20–25] and references therein). Noncommutative gauge theory defined through matrix models [26,27] contains a specific version of gravity as an intrinsic part, and provides a dynamical theory on noncommutative spaces. Noncommutative deformations of gravity have also led to a complex metric and gauge groups larger than the Lorentz group [20,22]. A noncommutative general relativity restricted to the volume-preserving transformations (unimodular theory of gravity) has been also constructed [28]. First steps toward a gauge theory of noncommutative gravity based on a θ-twisted approach have been made in [29]. Spherically symmetric spaces generated by four noncommutative coordinates in the frame formalism have also been investigated [30]. Lately, the version of noncommutative gravity obtained by the deformation of the diffeomorphism algebra [23] using the twist introduced in [12] has been most studied in the literature. However, it turned out that the dynamics of the noncommutative gravity arising from string theory [31] is much richer than this version of noncommutative gravity. The dynamics of closed strings in the presence of a constant B field induces a gravitational action in the next-to-leading order in the Seiberg-Witten limit [3]. Some of the three-graviton vertices have been derived and they cannot be obtained from an action written only in terms of the star product. It is suspected that the reason for this is the noncovariance of the Moyal star product under space-time diffeomorphisms. A geometrical approach to noncommutative gravity, leading to a general theory of noncommutative Riemann surfaces in which the problem of the frame dependence of the star product is also recognized, has been proposed in [32] (for further developments, see [33,34]).

A possibility of obtaining a theory, which is covariantly deformed under the local Poincaré transformations, is that of gauging the twisted Poincaré algebra itself. Einstein's theory of general relativity was formulated as a gauge theory of Lorentz symmetry by Utiyama [35] in 1956, while the Einstein-Cartan gravitational theory was formulated by Kibble [36] in 1961, as the gauge theory of Poincaré transformations. Instead of the partial derivatives in the Abelian twist element (1.3) one can use the covariant derivatives [36] (see also [37]):

$$\nabla_\mu = \partial_\mu + \frac{i}{2}\omega_\mu{}^{ab}\Sigma_{ab}, \tag{1.5}$$

where the (constant) spin matrices Σ_{ab} form a representation of the Lorentz algebra. We can define a covariant non-Abelian twist element as

$$\mathcal{T} = e^{-(i/2)\theta^{\mu\nu}\nabla_\mu\otimes\nabla_\nu + \mathcal{O}(\theta^2)}, \tag{1.6}$$

with possible covariant higher order terms in the noncommutativity parameter $\theta^{\mu\nu}$ in the exponent. In this paper we study the properties of such a covariant twist.

In Sec. II, we briefly review the commutative gauge theory of gravitation and in Sec. III, some basic aspects of twisting Hopf algebras are presented and the twisted Poincaré algebra is defined.

Section IV is devoted to the possibility of gauging the twisted Poincaré symmetry itself. A covariant non-Abelian twist element is defined by using the covariant derivative of the Poincaré gauge theory. The conditions ensuring that the Hopf algebra structure is preserved by the twist are verified. It is shown that the star product induced by the covariant twist is not associative. Therefore, the twisted Poincaré symmetry cannot be gauged by generalizing the Abelian twist (1.3) to a covariant non-Abelian twist (1.6), nor by introducing a more general covariant twist element.

II. COMMUTATIVE GAUGE THEORY OF GRAVITATION

General relativity (GR) still lacks the status of fundamental microscopic theory, because of the standing problems of quantization of the gravitational field and the existence of singular solutions under very general assumptions. Since the concept of gauge symmetry has been highly successful in describing the other three fundamental interactions, gauge theories of gravitation are very attractive. The important role of the Poincaré symmetry as the concept of relativistic invariance in the quantum field theory, leads one to consider the Poincaré gauge symmetry as a natural framework for describing the gravitational interaction.

The Einstein-Cartan theory of gravitation is a modification of GR, allowing space-time to have torsion, in addition to curvature, and relating torsion to the density of intrinsic angular momentum (the spin). In GR the Lorentz group, instead of the Poincaré group, is the structure group acting on the orthonormal Lorentz frames in the tangent spaces of the space-time manifold. Therefore, there is no room for translations in GR and thus for the torsion and spin tensors. In the Poincaré gauge theory, the torsion and its relation to the spin are naturally introduced, restoring the role of the Poincaré symmetry in relativistic gravity. Its geometric interpretation shows that the space-time has the structure of Riemann-Cartan geometry, possessing both *curvature* and *torsion* [36–40]. The curvature and torsion are surface densities of Lorentz transformations and translations, respectively.

The global Poincaré group is a ten-dimensional noncompact Lie group, which has the structure of a semidirect product of the translation group $\mathcal{T}_4$ and of the Lorentz group $SO(1,3)$, $\mathcal{P} = SO(1,3) \ltimes \mathcal{T}_4$. In order to define its transformations, we consider the Minkowski space-time $\mathcal{M}_4$, endowed with the real coordinates x^μ, $\mu = 0, 1, 2, 3$. The isometry group of $\mathcal{M}_4$ is the group of global Poincaré transformations, written in the infinitesimal form as

$$x'^\mu = x^\mu + \omega^\mu{}_\nu x^\nu + \epsilon^\mu, \qquad (2.1)$$

where $\omega^{\mu\nu} = -\omega^{\nu\mu}$ and ϵ^μ are the ten infinitesimal parameters associated to the Lorentz rotations and space-time translations, respectively.

In order to define matter fields on space-time (scalars, vectors, spinors, etc.), we consider the tangent space T_p at each point $p \in \mathcal{M}_4$. On each tangent space T_p we can use a coordinate frame (C), consisting of four vectors e_μ tangent to the coordinate lines, or a local Lorentz frame (L) of four orthonormal vectors $e_a(x)$,

$$e_a(x) \cdot e_b(x) = \eta_{ab} = \mathrm{diag}(-1, 1, 1, 1),$$

which are named the tetrad. The Latin indices $(a, b, \ldots)$ refer to the L frames and the Greek indices refer to the C frames. To each L frame $\{e_a\}$ we can associate local inertial coordinates x^a, $a = 0, 1, 2, 3$.

Now we consider the local Poincaré (gauge) group. In order to make the Lagrangian $L(\phi, \partial_a \phi)$ invariant under the local Poincaré transformations,

$$x'^a = x^a + \omega^a{}_b(x) x^b + \epsilon^a(x), \qquad (2.2a)$$

$$\phi'(x') = \left(1 - \frac{i}{2}\omega^{ab}(x)\Sigma_{ab}\right)\phi(x), \qquad (2.2b)$$

with the parameters $\omega^{ab}(x)$ and $\epsilon^a(x)$ depending on space-time coordinates, we have to introduce new compensating fields $e^a{}_\mu(x)$ and $\omega_\mu{}^{ab}(x) = -\omega_\mu{}^{ba}(x)$, named tetrads and spin connections, respectively [41]. They enable us to define the gauge-covariant derivative [35,36] (see also [37]), which in the C frame is written as

$$\nabla_\mu \phi = \left(\partial_\mu + \frac{i}{2}\omega_\mu{}^{ab}\Sigma_{ab}\right)\phi. \qquad (2.3)$$

The gauge fields $e^a{}_\mu(x)$ have inverses $e_a{}^\mu(x)$, which satisfy $e^a{}_\mu(x) e_b{}^\mu(x) = \delta^a_b$, $e^a{}_\mu(x) e_a{}^\nu(x) = \delta^\nu_\mu$. They can be used to transform C-frame indices $\mu, \nu, \ldots$ into the L-frame indices $a, b, \ldots$, and *vice versa*. Thus, we can define the covariant derivative with respect to the L frame by

$$\nabla_a \phi = e_a{}^\mu \nabla_\mu \phi. \qquad (2.4)$$

The quantities $R^{ab}{}_{\mu\nu}$ and $T^a{}_{\mu\nu}$,

$$R^{ab}{}_{\mu\nu} \equiv F^{ab}{}_{\mu\nu} = \partial_\mu \omega_\nu{}^{ab} - \partial_\nu \omega_\mu{}^{ab} + (\omega_\mu{}^{ac}\omega_\nu{}^{db} - \omega_\nu{}^{ac}\omega_\mu{}^{db})\eta_{cd}, \qquad (2.5)$$

$$T^a{}_{\mu\nu} \equiv F^a{}_{\mu\nu} = \partial_\mu e^a{}_\nu - \partial_\nu e^a{}_\mu + (\omega_\mu{}^{ab} e^c{}_\nu - \omega_\nu{}^{ab} e^c{}_\mu)\eta_{bc}, \qquad (2.6)$$

obtained from the commutator of two covariant derivatives, $[\nabla_a, \nabla_b]\phi = (\frac{1}{2}F^{cd}{}_{ab}\Sigma_{cd} - F^c{}_{ab}\nabla_c)\phi$, are identified with the components of the curvature and torsion tensors of the space-time, respectively. Therefore, the Poincaré gauge theory of gravitation has the geometric structure of the Riemann-Cartan space $\mathcal{U}_4$ with curvature and torsion.

The metric tensor can be defined by using the tetrad gauge fields. In a C frame it has the components

$$g_{\mu\nu}(x) = \eta_{ab}e^a{}_\mu(x)e^b{}_\nu(x). \tag{2.7}$$

According to (2.7) the metric itself can be seen as an effective gauge field, i.e., a dynamical variable.

By imposing the condition of null torsion, $T^a{}_{\mu\nu} = 0$, one can solve for the spin connection $\omega_\mu{}^{ab}$ in terms of the tetrads $e^a{}_\mu$, thus reducing the Einstein-Cartan theory to GR.

III. TWISTING THE POINCARÉ ALGEBRA

To describe physics on the noncommutative space-time generated by (1.2), one replaces the usual pointwise product of functions, $f(x)$ and $g(x)$, by the noncommutative Moyal star product:

$$\begin{aligned}(f \star g)(x) &= f(x)\exp\left(\frac{i}{2}\overleftarrow{\partial}_\mu\theta^{\mu\nu}\overrightarrow{\partial}_\nu\right)g(x)\\ &= f(x)g(x) + \sum_{n=1}^{\infty}\frac{1}{n!}\left(\frac{i}{2}\right)^n\theta^{\mu_1\nu_1}\cdots\theta^{\mu_n\nu_n}\\ &\quad\times(\partial_{\mu_1}\cdots\partial_{\mu_n}f(x))(\partial_{\nu_1}\cdots\partial_{\nu_n}g(x)).\end{aligned} \tag{3.1}$$

The noncommutative space-time does not possess Lorentz symmetry when $\theta_{\mu\nu}$ in (1.2) is a constant antisymmetric matrix. This could be a serious problem, because the quantum and gauge field theories of high energy physics are vitally dependent on the representation content of the Poincaré algebra. The solution to the problems arising from the breaking of the Lorentz symmetry is the twisted Poincaré symmetry [12,13].

In the language of quantum groups, one can deform a cocommutative Hopf algebra like the universal enveloping algebra of a Lie algebra $\mathcal{G}$, denoted below by $\mathcal{U}(\mathcal{G})$, to a non-cocommutative one by introducing a twist element, $\mathcal{F} \in \mathcal{U}(\mathcal{G}) \otimes \mathcal{U}(\mathcal{G})$, which modifies the coproduct of the Hopf algebra by a similarity transformation

$$\Delta_0(X) \to \Delta_t(X) = \mathcal{F}\Delta_0(X)\mathcal{F}^{-1}, \qquad X \in \mathcal{G}, \tag{3.2}$$

in other words, by twisting the coproduct of $\mathcal{U}(\mathcal{G})$ [42] (see also the monographs [43]). In order to preserve the Hopf algebra structure, the twist element has to satisfy the twist condition

$$\mathcal{F}_{12}(\Delta_0 \otimes \mathrm{id})\mathcal{F} = \mathcal{F}_{23}(\mathrm{id} \otimes \Delta_0)\mathcal{F}, \tag{3.3}$$

where $\mathcal{F}_{12} = \mathcal{F} \otimes \mathbf{1}$ and $\mathcal{F}_{23} = \mathbf{1} \otimes \mathcal{F}$. The twisting of the coproduct (3.2) is accompanied by a modification of the product in the algebra of representation of $\mathcal{U}(\mathcal{G})$ as in (1.4). The twist element does not affect the multiplication of the generators of the Lie algebra and therefore the commutation relations among the generators of $\mathcal{U}(\mathcal{G})$ are preserved. This means that the representation content of the twisted Hopf algebra $\mathcal{U}_t(\mathcal{G})$ is identical with that of $\mathcal{U}(\mathcal{G})$.

In the framework sketched above, the Poincaré algebra $\mathcal{P}$ has a commutative subalgebra of translation generators $P_\mu = -i\partial_\mu$ that can be used to construct the *Abelian* twist element

$$\mathcal{F} = e^{(i/2)\theta^{\mu\nu}P_\mu \otimes P_\nu}, \tag{3.4}$$

where $\theta^{\mu\nu}$ is a real constant antisymmetric matrix. This element satisfies the twist condition (3.3) and thus it can be used to consistently twist the coproduct of the Hopf algebra $\mathcal{U}(\mathcal{P})$.

Since the Abelian twist element (3.4) only involves the generators P_μ, only the coordinate dependency of the fields $\phi(x)$ is involved in the deformed multiplication of the fields. Therefore, the matrix-valued generators $\Sigma_{\mu\nu}$ act on the component degrees of freedom of the fields $\phi(x)$ in the same way, in the deformed and nondeformed algebra cases, i.e., through the matrix multiplication and the symmetric coproduct

$$\Delta_t(\Sigma_{\mu\nu}) = \Delta_0(\Sigma_{\mu\nu}) = \mathbf{1} \otimes \Sigma_{\mu\nu} + \Sigma_{\mu\nu} \otimes \mathbf{1}. \tag{3.5}$$

It should, however, be mentioned that the definition of fields on noncommutative space-time is more involved than in the commutative theory [14,15].

The noncommutative quantum field theories built through Weyl quantization and the canonical star product (3.1) possess the twisted Poincaré symmetry, which represents the concept of relativistic invariance in noncommutative field theories. This also enables us to adopt the point of view according to which the noncommutativity of coordinates (1.2) is required by the twisted Poincaré symmetry of space-time.

IV. GAUGING THE TWISTED POINCARÉ SYMMETRY

The local Poincaré symmetry is an *external* gauge symmetry. Through geometrical interpretation the Poincaré gauge symmetry translates to the covariance under general coordinate transformations and to the local Lorentz symmetry. This ‘‘duality’’ of the Poincaré gauge symmetry is both a problem and a possibility, since it has been shown that an internal gauge symmetry cannot be twisted together with the Poincaré symmetry [16,17]. We can attempt to gauge the twisted Poincaré algebra itself and find out whether the gauge theory of the Poincaré symmetry on noncommutative space-time can be formulated by means of a gauge-covariant twist.

We could take the direct naive approach and try to construct a noncommutative gauge theory of the twisted

Poincaré symmetry by using the Abelian twist (3.4) and by replacing the pointwise product of functions with the Moyal star product in the classical theory constructed in Sec. II. The result would, however, be an inconsistent frame-dependent theory (due to the frame dependence of the star product)—in many ways similar to those already developed—which cannot be a plausible theory of gravitation. We would not be able to give any meaningful geometrical interpretation to a theory of this type.

Since the global Poincaré symmetry is twisted with the Abelian twist (3.4) in the case of the flat noncommutative space-time, also the generalized Poincaré gauge symmetry on noncommutative space-time should be a quantum symmetry. A natural way to generalize the Poincaré gauge symmetry to the noncommutative setting is to consider it as a twisted gauge symmetry, so that the global twisted Poincaré symmetry is obtained in the limit of vanishing gauge fields. When the global twisted Poincaré symmetry is generalized to a gauge symmetry, we have to introduce the gauge fields in order to compensate the noncovariance of the partial derivatives, similarly as in the commutative case. Partial derivatives ∂_μ will be replaced by covariant derivatives, which in the coordinate frame read

$$\nabla_\mu = d_\mu + \mathcal{A}_\mu(x) = i\Big(e^a{}_\mu(x)P_a + \tfrac{1}{2}\omega_\mu{}^{ab}(x)\Sigma_{ab}\Big), \tag{4.1}$$

where the Σ_{ab} generate a finite-dimensional representation of the Lorentz algebra. The difference compared to the covariant derivative of an internal gauge symmetry [17]

$$D_\mu = \partial_\mu + iA_\mu(x) = i(P_\mu + A^a_\mu(x)T_a) \tag{4.2}$$

are the tetrad gauge fields $e^a{}_\mu$ multiplying P_a in (4.1). $\mathcal{A}_\mu$ are the gauge fields associated to the local Lorentz transformations. In order to obtain a theory that is covariantly deformed under the Poincaré gauge transformations, the frame-dependent translation generators P_a have to be replaced by the covariant derivatives $-i\nabla_\mu$ in the Abelian twist element (3.4). The covariant non-Abelian twist element is of the form

$$\mathcal{T} = e^{-(i/2)\theta^{\mu\nu}\nabla_\mu\otimes\nabla_\nu + \mathcal{O}(\theta^2)}, \tag{4.3}$$

where $\mathcal{O}(\theta^2)$ stands for the possible additional covariant terms in higher orders of the noncommutativity parameter $\theta^{\mu\nu}$.[1] Because of the similar forms of the covariant derivatives (4.1) and (4.2) and of their twist elements, the basic algebraic reasoning presented in [17] holds also for the twist element (4.3) proposed here. The gauge fields $\mathcal{A}_\mu$ alone in ∇_μ will violate the twist condition (3.3) and the rest of gauge fields $e^a{}_\mu$are not able to rescue the twist condition. The fact that there are now two second rank (field strength) tensors (2.5) and (2.6) does not help to satisfy the twist condition.

Following the arguments of [17], we can attempt to impose the twist condition (3.3). First we consider the twist element (4.3) with only the first order term in θ in the exponent. The second order terms in θ that do not cancel in the twist condition (3.3) are, in the left-hand side

$$\frac{1}{2}\Big(-\frac{i}{2}\Big)^2\theta^{\mu\nu}\theta^{\rho\sigma}(2\nabla_\mu\nabla_\rho\otimes\nabla_\nu\otimes\nabla_\sigma + 2\nabla_\mu\otimes\nabla_\nu\nabla_\rho \otimes\nabla_\sigma + \nabla_\mu\otimes\nabla_\rho\otimes\nabla_\nu\nabla_\sigma + \nabla_\rho\otimes\nabla_\mu\otimes\nabla_\nu\nabla_\sigma) \tag{4.4}$$

and in the right-hand side

$$\frac{1}{2}\Big(-\frac{i}{2}\Big)^2\theta^{\mu\nu}\theta^{\rho\sigma}(2\nabla_\rho\otimes\nabla_\mu\nabla_\sigma\otimes\nabla_\nu + 2\nabla_\rho\otimes\nabla_\mu \otimes\nabla_\nu\nabla_\sigma + \nabla_\mu\nabla_\rho\otimes\nabla_\nu\otimes\nabla_\sigma + \nabla_\mu\nabla_\rho\otimes\nabla_\sigma\otimes\nabla_\nu). \tag{4.5}$$

These terms cannot be canceled by terms that contain second rank tensors

$$R^{ab}{}_{\mu\nu}\Sigma_{ab}, \qquad T^a{}_{\mu\nu}\nabla_a, \tag{4.6}$$

because the two indices for such tensors come from the same $\theta^{\mu\nu}$, unlike for the $\nabla\nabla$ factors in (4.4) and (4.5). This is why such terms are not included in twist element (4.3) in the first place. The other possible second order terms in (4.3) have the forms

$$\theta^{\mu\nu}\theta^{\rho\sigma}1\otimes\nabla_\mu\nabla_\nu\nabla_\rho\nabla_\sigma, \qquad \theta^{\mu\nu}\theta^{\rho\sigma}\nabla_\mu\nabla_\nu\nabla_\rho\nabla_\sigma\otimes 1, \tag{4.7}$$

$$\theta^{\mu\nu}\theta^{\rho\sigma}\nabla_\mu\otimes\nabla_\nu\nabla_\rho\nabla_\sigma, \qquad \theta^{\mu\nu}\theta^{\rho\sigma}\nabla_\mu\nabla_\nu\nabla_\rho\otimes\nabla_\sigma, \tag{4.8}$$

$$\theta^{\mu\nu}\theta^{\rho\sigma}\nabla_\mu\nabla_\nu\otimes\nabla_\rho\nabla_\sigma, \tag{4.9}$$

with all the permutations of indices of the covariant derivatives—although the antisymmetry of θ greatly reduces the number of independent permutations. We have verified that when introduced into the twist element (4.3) and consequently into the twist condition (3.3), these second orders terms can never cancel all the terms in (4.4) and (4.5). Therefore, the twist condition (3.3) cannot be fulfilled in the second order in θ.

It is well known that the gauging of the translation symmetry leads to the Einstein-Hilbert Lagrangian and to the covariance under general coordinate transformations [37]. Hence, it is interesting to see whether the gauge theory of the *external* translation symmetry group $\mathcal{T}_4$ can be consistently defined together with the twisted Poincaré symmetry. The covariant derivative for the local translations is

[1]The following discussion is presented for the exponential form (4.3), but the results are valid for any invertible functional form.

$$d_\mu = ie^a{}_\mu P_a. \tag{4.10}$$

In fact, this is also the covariant derivative of the Poincaré gauge symmetry for one-dimensional representations, for which the covariant derivative (4.1) should reduce to (4.10). Clearly the gauge fields $e^a{}_\mu$ now contain contributions also from the local Lorentz transformations. Since the covariant derivatives of the translation group do not commute,

$$\begin{aligned} [d_\mu, d_\nu] &= C^\rho{}_{\mu\nu} d_\rho, \\ C^\rho{}_{\mu\nu} &= (e^a{}_\mu \partial_a e^b{}_\nu - e^a{}_\nu \partial_a e^b{}_\mu) e_b{}^\rho, \end{aligned} \tag{4.11}$$

the covariant element

$$\mathcal{T} = e^{-(i/2)\theta^{\mu\nu} d_\mu \otimes d_\nu + \mathcal{O}(\theta^2)} = e^{i/2\theta^{\mu\nu} e^a{}_\mu P_a \otimes e^b{}_\nu P_b + \mathcal{O}(\theta^2)} \tag{4.12}$$

cannot be of the Abelian type (3.4), which is known to be a twist. Because of this and the high level of arbitrariness in choosing the gauge fields $e^a{}_\mu$ in the translationally covariant twist (4.12), we face similar algebraic problems as with the covariant twist element (4.3) of the full Poincaré gauge symmetry. The twist element (4.12) does not satisfy the twist condition (3.3), even though its form is much simpler now. Thus, it is not only the local Lorentz symmetry that breaks the validity of the non-Abelian Poincaré gauge-covariant twist element (4.3); the external gauge symmetry associated with the general coordinate transformations is just as problematic.

Thus, we have obtained the result that the Poincaré gauge-covariant non-Abelian element (4.3) is not a twist and the star product defined by it is not associative. We can conclude that *the twisted Poincaré symmetry cannot be gauged by generalizing the Abelian twist* (3.4) *to a covariant non-Abelian twist* (4.3), *nor by introducing a more general covariant twist element.*

It should be mentioned that from the mathematical point of view, we could try to deform the action of the twisted Poincaré algebra on its representations, instead of generalizing the twist element, but it seems unlikely that such an approach could solve the problems related to the frame-dependent twist element (3.4).

V. CONCLUDING REMARKS AND PERSPECTIVES

In this paper we have investigated the possibility of gauging the twisted Poincaré symmetry in order to obtain a noncommutative gauge theory of gravitation. A covariant non-Abelian twist element $\mathcal{T}$ has been defined by using the covariant derivative of the commutative Poincaré gauge theory. The twist condition that assures the associativity of the multiplication of the representations of the twisted Poincaré algebra is violated already in the second order in the noncommutativity parameter $\theta^{\mu\nu}$. Adding gauge-covariant terms of higher orders in $\theta^{\mu\nu}$ into the definition of the twist $\mathcal{T}$ does not improve the result. When we restrict the gauge symmetry to the translation group $\mathcal{T}_4$, we are faced with similar algebraic problems as in the case of the full Poincaré symmetry. Thus, both the local Lorentz symmetry and the local translational symmetry, associated with the covariance under general coordinate transformations, violate the twist condition already in the second order in the parameter $\theta^{\mu\nu}$.

The question of unifying the external (global or local) Poincaré symmetry and the internal gauge symmetry under a common twist remains an open fundamental problem of noncommutative gauge theories.

Since the introduction of a gauge-covariant twist breaks the associativity of the algebra of functions on noncommutative space-time, both in the internal and external gauge symmetry cases, we may have to consider space-time geometries that are also nonassociative, not only noncommutative. Indeed, there exist in the literature works on constructing nonassociative theories with some desired properties (see, e.g., [44–46] and references therein). There are, as well, attempts to retain the associativity of the star products defined with covariant derivatives, by imposing appropriate constraints on Poisson manifolds (see [47] and references therein).

The nonassociativity, as well as the noncommutativity, has its origin in string theory. It is known that in the presence of a constant background field, $\omega = B + F$, the noncommutative geometry is described by the Moyal product, which is associative [3] (see also [48,49]). The physics of this case, corresponding to a flat brane embedded in a flat background space, is well understood [50]. When ω is not constant, but it satisfies $d\omega = 0$, the target space becomes a Poisson manifold and thus the Kontsevich prescription [51] can be used to define the associative product. In the most general case, $d\omega \neq 0$, it has been established that the extension of the Kontsevich as well as the Moyal products become nonassociative [48].

Defining gauge theory on nonassociative manifolds is not straightforward. Recently, there have been attempts to restore the associativity of the star product, when the ordinary derivative is replaced by the covariant derivative. In [52] (see also [24] for details and example) it was considered a curved background and a $\theta^{\mu\nu}$, which is a covariantly constant antisymmetric tensor, $D_\mu \theta^{\nu\rho} = 0$. However, the appearance of the commutators $[D_\mu, D_\nu]$ in the star product, which vanish only on scalar functions, spoils the associativity. By considering a background space endowed with a Friedmann-Robertson-Walker metric, it was found [52] that $\theta^{\mu\nu}$ can be chosen such that the nonassociativity appears at the fourth order in θ, while the noncommutative effects are already present starting with θ^2. No extension to a "complete" associative star product has, however, been obtained.

In [49] it was suggested that such a nonassocitivity "anomaly" can be removed by including the Chan-Paton factors to define the associative star product, starting from the axioms of the rational conformal field theory. It is argued that by using the vacuum string field theory, one may push most of the D branes in a so-called ''closed string vacuum.'' In this case the associativity is restored, i.e., the Chan-Paton factors modify the originally nonassociative algebra to an associative one. An infinite number of D branes are, however, needed for this modification.

In the gauge theory of the twisted Poincaré algebra proposed in our work, the twist condition (3.3) is not satisfied. This means that the algebra of the twist symmetry does not close, a property that also implies the nonassociativity of the star product.

We believe, however, that in formulating the gauge theory of noncommutative gravity, the requirement of general coordinate transformations with respect to the whole Lorentz group should be relaxed and replaced by the requirement of general coordinate transformations only under the residual symmetry of the noncommutative field theories as argued in [15]. This approach will be pursued in a forthcoming communication [53]. In a quite different context, the description of nature at the Planck scale is suggested to be given by a nonlocal translationally invariant theory, the so-called ''very special relativity,'' with a symmetry under a subgroup of the Lorentz group [54], while at the low-energy scale the Poincaré invariance would be operating. A realization of such a symmetry, for the Planck scale part, has been recently given [55] on the noncommutative space-time with lightlike noncommutativity. A gauge theory of the latter symmetry can be performed as mentioned above [53].

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