Gauge Model Based on Group $G \times SU(2)$ *

ZET Gheorghe[1**], MANTA Vasile[2], POPA Camelia[3]

[1] *Department of Physics, Gh. Asachi Technical University, Iasi, Romania*

[2] *Department of Computers, Gh. Asachi Technical University, Iasi, Romania*

[3] *Faculty of Physics, Al. I. Cuza University, Iasi, Romania*

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We present a model of gauge theory based on the symmetry group $G \times SU(2)$ where G is the gravitational gauge group and $SU(2)$ is the internal group of symmetry. We employ the spacetime of four-dimensional Minkowski, endowed with spherical coordinates, and describe the gauge fields by gauge potentials. The corresponding strength field tensors are calculated and the field equations are written. A solution of these equations is obtained for the case that the gauge potentials have a particular form with spherical symmetry. The solution for the gravitational potentials induces a metric of Schwarzschild type on the gravitational gauge group space.

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Gauge theories are fundamental in the field theories and, in particular, in elementary particle physics.[1,18,19,21] The three non-gravitational interactions (electromagnetic, weak and strong) are completely described by means of gauge theory in the framework of the standard model. The gauge group is chosen as a direct product $SU(3) \times SU(2) \times U(1)$ and it is considered as a local group of symmetry. On the other hand, the gauge group for the gravitational interaction is the Poincaré group,[6,11,13] or the de-Sitter group if we introduce the cosmological constant into the model.[22] A more elaborated model of superstring theory also includes the gauge theory and is considered as an adequate framework to describe all the four fundamental interactions (electromagnetic, weak, strong and gravitational).

In this Letter, we develop a model of gauge theory which combines the internal and gravitational local symmetries. We use the quantum gauge theory of gravitation developed by Wu.[15,16,17] We present this gauge theory by choosing $G \times SU(2)$ as the local group of symmetry. It is the direct product of the gravitational gauge group G (Ref. [16]) with the internal group $SU(2)$. In this theory the gravitational field is treated as a physical interaction in a Minkowski (flat) spacetime M,[20] and all gauge fields are described by gauge potentials.

We obtain solutions for field equations corresponding to the case when the gauge potentials have spherical symmetry. The spacetime is a four-dimensional Minkowski one, endowed with spherical coordinates, and the gauge potentials are chosen in a particular spherical symmetric form. We give some concluding remarks and few open questions in the unified gauge models.

We consider a gauge theory based on the local symmetry group $G \times SU(2)$, which is the direct product of the gravitational gauge group G and the internal group $SU(2)$.[1−9] In this theory the gravitation is treated as a physical interaction in a Minkowski (flat) spacetime M, and the gravitational filed is described by gauge potentials. The generators of the gravitational gauge group G are denoted by P_α, $\alpha = 1, 2, 3, 0$, and they have expression $P_\alpha = -i\partial_\alpha$ as differential operators. They are commuting operators

$$[P_\alpha, P_\beta] = 0. \tag{1}$$

The generators of $SU(2)$ group are denoted by T_a, $a = 1, 2, 3$, and their commutation relations are

$$[T_a, T_b] = f^c_{ab} T_c, \tag{2}$$

where $f^c_{ab} = -f^c_{ba}$ are the structure constants of $SU(2)$ group and they coincide with the total anti-symmetric Levi-Civita symbol of third rank ε_{abc} with $\varepsilon_{123} = +1$. Because of the direct-product structure of the gauge group $G \times SU(2)$, we have also $[T_a, P_\alpha] = 0$.

As usual, we introduce the gauge vector field $C_\mu(x)$ with values in the Lie algebra of the group G,

$$C_\mu(x) = C^\alpha_\mu(x) P_\alpha, \quad \mu = 1, 2, 3, 0, \tag{3}$$

where $C^\alpha_\mu(x)$ are the gravitational gauge potentials. The corresponding gauge covariant derivative is defined as

$$D_\mu = \partial_\mu - igC_\mu(x), \tag{4}$$

where g is the gauge coupling constant of the gravitational interactions. The corresponding strength field tensor $F_{\mu\nu}(x) = F^\alpha_{\mu\nu}(x) P_\alpha$, with values in the Lie algebra of G, has the components[16,22]

$$F^\alpha_{\mu\nu} = G^\beta_\mu \partial_\beta C^\alpha_\nu - G^\beta_\nu \partial_\beta C^\alpha_\mu, \tag{5}$$

where

$$G^\alpha_\mu(x) = \delta^\alpha_\mu - gC^\alpha_\mu(x) \tag{6}$$

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**Email: gzet@phys.tuiasi.ro

represents the new gravitational gauge potentials. We suppose that these new potentials admit the inverses $\bar{G}^{\mu}_{\alpha}(x)$ with the usual properties

$$\bar{G}^{\mu}_{\alpha}G^{\beta}_{\mu} = \delta^{\beta}_{\alpha}, \quad \bar{G}^{\mu}_{\alpha}G^{\alpha}_{\nu} = \delta^{\mu}_{\nu}. \tag{7}$$

It is easy to verify that the gauge derivatives (4) can be written in the form

$$D_\mu = G^{\beta}_{\mu}\partial_\beta. \tag{8}$$

Following Refs. [16,22], we define a metric tensor on the gravitational gauge group manifold by

$$g_{\alpha\beta} = \eta_{\mu\nu}\bar{G}^{\mu}_{\alpha}\bar{G}^{\nu}_{\beta}, \quad g^{\alpha\beta} = \eta^{\mu\nu}G^{\alpha}_{\mu}G^{\beta}_{\nu}, \tag{9}$$

where $\eta_{\mu\nu} = \mathrm{diag}(1,1,1,-1)$ is the metric tensor of the Minkowski spacetime M and $\eta^{\mu\nu}$ denotes its inverse.

Analogously, the $SU(2)$ internal gauge potentials $A^{a}_{\mu}(x)$ are defined by the formula

$$A_\mu(x) = A^{a}_{\mu}(x)T_a. \tag{10}$$

The strength tensor field $A_{\mu\nu}(x) = A^{a}_{\mu\nu}T_a$ of the gauge potentials $A^{a}_{\mu}(x)$ has the components

$$A^{a}_{\mu\nu}(x) = D_\mu A^{a}_{\nu} - D_\nu A^{a}_{\mu} + g_1 f^{a}_{bc}A^{b}_{\mu}A^{c}_{\nu}, \tag{11}$$

where g_1 is the $SU(2)$ gauge coupling constant. Using Eq. (8), we can write the components (11) as

$$A^{a}_{\mu\nu}(x) = G^{\alpha}_{\mu}\partial_\alpha A^{a}_{\nu} - G^{\alpha}_{\nu}\partial_\alpha A^{a}_{\mu} + g_1 f^{a}_{bc}A^{b}_{\mu}A^{c}_{\nu}. \tag{12}$$

The tensor $A^{a}_{\mu\nu}(x)$ is not a gauge covariant one, and therefore we have to introduce its covariant version with the components[16]

$$\mathbb{A}^{a}_{\mu\nu}(x) = A^{a}_{\mu\nu}(x) + g\bar{G}^{\lambda}_{\alpha}A^{a}_{\lambda}F^{\alpha}_{\mu\nu}. \tag{13}$$

The field equations for the gravitational gauge field $C^{\alpha}_{\mu}(x)$ are[16,22]

$$\begin{aligned}\partial_\mu\Big(&\frac{1}{4}\eta^{\mu\rho}\eta^{\nu\sigma}g_{\alpha\beta}F^{\beta}_{\rho\sigma} - \frac{1}{4}\eta^{\nu\rho}F^{\mu}_{\rho\alpha} + \frac{1}{4}\eta^{\mu\rho}F^{\nu}_{\rho\alpha}\\ &- \frac{1}{2}\eta^{\mu\rho}\delta^{\nu}_{\alpha}F^{\beta}_{\rho\beta} + \frac{1}{2}\eta^{\nu\rho}\delta^{\mu}_{\alpha}F^{\beta}_{\rho\beta}\Big) = -gT^{\nu}_{\alpha},\end{aligned} \tag{14}$$

where T^{ν}_{α} is the gravitational energy-momentum tensor. [16] On the other hand, the field equations of the $SU(2)$ internal gauge field $A^{a}_{\mu\nu}(x)$ are

$$\partial^\mu\mathbb{A}^{a}_{\mu\nu} = -g_1\eta_{\nu\sigma}J^{\sigma}_{a}, \tag{15}$$

where J^{σ}_{a} $(a = 1,2,3)$ are the corresponding conserved currents[12,14]

$$\partial_\mu J^{\mu}_{a} = 0. \tag{16}$$

We will obtain a solution of the field equations (14) and (15) for the case of the spherical symmetry.

As an application, we will consider a $G \times SU(2)$ gauge theory with spherical symmetry. The spacetime is a four-dimensional Minkowski endowed with spherical coordinates:

$$ds^2 = dr^2 + r^2d\theta^2 + r^2\sin^2\theta\, d\phi^2 - dt^2. \tag{17}$$

For the gravitational gauge potentials we choose a particular form with the following non-null components

$$\begin{aligned}&C^{r}_{r} = U(r), \qquad C^{\theta}_{\theta} = \frac{r-1}{rg},\\ &C^{\phi}_{\phi} = \frac{r\sin\theta - 1}{rg\sin\theta}, \quad C^{t}_{t} = -\frac{U(r)}{1-gU(r)},\end{aligned} \tag{18}$$

where U is a function depending only of the 3D radius r. For the $SU(2)$ internal gauge potentials $A^{a}_{\mu}(x)$ we consider only one non-null component

$$A^{3}_{t} = V(r), \tag{19}$$

with V a function only of the variable r. Then the field equations (14) and (15) become

$$\begin{aligned}(1-gU)(rV'' + 2V') &- grVU'' - 2gVU'\\ &- 2grU'V' = 0,\end{aligned} \tag{20}$$

for $SU(2)$ gauge fields, and respectively

$$2grU'(1-gU) + 2U - gU^2 = 0, \tag{21}$$

for the gravitational gauge field. In fact, for the gauge gravitational field one obtains four field equations, but they are equivalent and this is a correct result because we have only an unknown function $U(r)$.

The general solution of Eq. (21) is

$$U(r) = \frac{1 \pm \sqrt{1 + \dfrac{a}{r}}}{g}, \tag{22}$$

where a is an arbitrary constant of integration. If we choose $a = -2Gm$, where m is supposed to be the mass of the point-like source of our gravitational field, then the result (22) corresponds to a Schwarzschild type solution having the square of the line element [see the definition (9) of the metric coefficients] equal to

$$\begin{aligned}ds^2 = \frac{dr^2}{1 - \dfrac{2Gm}{r}} &+ r^2(d\theta^2 + \sin^2\theta d\varphi^2)\\ &- \Big(1 - \frac{2Gm}{r}\Big)dt^2.\end{aligned} \tag{23}$$

We remember that this line element is defined on the gravitational group manifold and that the spacetime remains a Minkowski (flat) one. As it is shown in Ref. [15], the quantum gauge theory of gravity imposes the relationship $g^2 = 4\pi G$ between the gauge coupling constant g of the gravitational interactions and the Newton's gravitational constant G. Therefore, the Schwarzschild solution (23) has a dependence of the gauge coupling constant g, a result that is in accord with both quantum gauge theory of gravity and general relativity.

Now, if we introduce the solution (22) in Eq. (20), then we obtain the following two possible solutions for the $SU(2)$ gauge potential $V(r)$

$$V_1(r) = \sqrt{1 + \frac{a}{r}}, \quad V_2(r) = \sqrt{\frac{r}{r+a}}, \tag{24}$$

corresponding to the sign "+" and respectively "−" from the solution (22). The solution $V_1(r)$ has a singularity in the origin $r = 0$, while the second one $V_2(r)$ is finite in the origin $r = 0$, but it has a singularity for $r = -a$. When $r \to \infty$, both solutions are finite and we have $V_{1,2}(r) \to 1$.

We also observe that for the spherical model considered here, there are only gravitational couplings between fields, but not internal $SU(2)$ gauge couplings, because the corresponding solutions do not include the coupling constant g_1.

In summary, we have constructed a model of gauge theory using the quantum gauge theory of gravity developed by Wu.[15,16,17] In this theory, the gravitational interaction is considered as a fundamental interaction in a flat Minkowski spacetime, and all the fields (both internal and gravitational) are represented by gauge potentials.

The strength field tensors of the gravitational and internal gauge potentials have been calculated, and the corresponding field equations have been written. For the case of internal strength field tensor, associated to the $SU(2)$ group, we obtain a covariant expression by using a gauge covariant derivative which includes not only the internal but also the gravitational gauge potentials. The field equations have on their right-hand sides the gravitational energy-momentum tensor [see Eq. (14)] and the internal $SU(2)$ currents [see Eq. (16)] which are conserved quantities.

As an example of application of the developed $G \times SU(2)$ model, we considered the case when the gauge fields have spherical symmetry. The corresponding field equations have been written and their solutions obtained. The solution for the gravitational gauge potentials induces a metric of Schwarzschild type [see Eq. (23)] on the gauge gravitational group manifold. However, the spacetime remains a Minkowski (flat) one, endowed with spherical coordinates.

We have obtained two solutions for the internal $SU(2)$ gauge potentials: $V_1(r)$ and $V_2(r)$ [see Eq. (24)]. The solution $V_1(r)$ has a singularity in the origin $r = 0$, while the second one $V_2(r)$ is finite in the origin $r = 0$, but it has a singularity for $r = -a$. When $r \to \infty$, both solutions are finite and we have $V_{1,2}(r) \to 1$.

Our results show that for the particular spherical model considered, there are only gravitational couplings between fields, but not internal $SU(2)$ gauge couplings, because the corresponding solutions do not include the $SU(2)$ coupling constant g_1.

Because the spacetime used in our model of gauge theory remains Minkowski (flat), i.e., it is not affected by gravitation, the quantization of the gravitational field can be obtained by the path integral method on a similar way with that from internal gauge models. In addition, the Poincaré group is considered as a purely inner symmetry and this assures the renormalizability property of our unified gauge model.[13,16]

The gauge model developed can be generalized to an arbitrary gauge group. All fields have to be represented by gauge potentials and the spacetime remains a Minkowski (flat) one, even if we introduce the gravitation into the model. Such a formulation could lead to a consistent quantum theory of gravity.[15] These aspects remain as open questions for the future research.

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