Available online at www.sciencedirect.com
ScienceDirect
PHYSICS LETTERS B
Physics Letters B 660 (2008) 573–578
www.elsevier.com/locate/physletb

Corrections to Schwarzschild solution in noncommutative gauge theory of gravity

M. Chaichian [a], A. Tureanu [a,*], G. Zet [b]

[a] *Department of Physics, University of Helsinki and Helsinki Institute of Physics, PO Box 64, FIN-00014 Helsinki, Finland*
[b] *Department of Physics, "Gh. Asachi" Technical University, Bd. D. Mangeron 67, 700050 Iasi, Romania*

Received 10 October 2007; received in revised form 14 January 2008; accepted 19 January 2008

Available online 25 January 2008

Editor: L. Alvarez-Gaumé

Abstract

A deformed Schwarzschild solution in noncommutative gauge theory of gravitation is obtained. The gauge potentials (tetrad fields) are determined up to the second order in the noncommutativity parameters $\Theta^{\mu\nu}$. A deformed real metric is defined and its components are obtained. The noncommutativity correction to the red shift test of general relativity is calculated and it is concluded that the correction is too small to have observable effects. Implications of such a deformed Schwarzschild metric are also mentioned.

1. Introduction

The noncommutativity of space–time is a compelling option for the description of quantized space–time and its study is significant for answering the ultimate question about the quantum nature of space–time at very high energy scales. If nature has chosen such a course, it is most sensible to search for manifestations of the noncommutativity of space–time at the "natural laboratories" of the highest energy, i.e., the gravitational singularities.

The noncommutativity of space–time is intrinsically connected with gravity [1,2]. Gauge theories of gravitation have been intensively studied up to now, both on commutative [3,4] (see also the reviews [5,6]) [7] and noncommutative [8,9] space–times. Many recent investigations are oriented towards a formulation of general relativity on noncommutative space–times. In Ref. [8] for example, a deformation of Einstein's gravity was studied by gauging the noncommutative $SO(4,1)$ de Sitter group and using the Seiberg–Witten map [2,10,11] with subsequent contraction to the Poincaré (inhomogeneous Lorentz) group $ISO(3,1)$. Another construction of noncommutative gravitational theory, based on the twisted Poincaré algebra [12] was proposed in Ref. [13]. The twisting procedure insures the invariance of the algebra $[x^\mu, x^\nu] = i\Theta^{\mu\nu}$ (canonical structure) defining the noncommutativity of the space–time; however, it turned out that the dynamics of the noncommutative gravity coming from string theory [14] is much richer than the one in this version of deformed gravity [13]. In Ref. [15] a noncommutative version of general relativity was proposed for a restrictive class of coordinate transformations which preserve the canonical structure. By gauging the Lorentz algebra $so(3,1)$ within the enveloping algebra approach one obtains a theory of noncommutative general relativity restricted to the volume-preserving transformations (unimodular theory of gravity). Another attempted approach was to twist the gauge Poincaré algebra [16]. It is worthwhile to emphasize that there remains one more important unsolved problem in all these theories: to establish a Leibniz rule for gauge transformations of fields [17,18], since the star product is not invariant under the diffeomorphism transformations. Steps towards this goal have been taken in a geometrical approach to noncommutative gravity [19].

* Corresponding author.
E-mail address: anca.tureanu@helsinki.fi (A. Tureanu).

0370-2693/$ – see front matter
doi:10.1016/j.physletb.2008.01.029

In this Letter, proceeding along the approach in Ref. [8], we present a deformed Schwarzschild solution in noncommutative gauge theory of gravitation. Although this version of noncommutative gravity is certainly not a final one, we believe that the complete theory will retain the main features of this approach. First, we recall the results of a previous study, in which a de Sitter gauge theory of gravitation over a spherically symmetric commutative Minkowski space–time was developed [7]. Then, a deformation of the gravitational field is constructed by gauging the noncommutative de Sitter $SO(4,1)$ group [8] and using Seiberg–Witten map [2]. The space–time of noncommutative theory will be also of Minkowski type but it will be endowed with spherical noncommutative coordinates. The deformed gauge fields are determined up to the second order in the noncommutativity parameters $\Theta^{\mu\nu}$.

Finally, the deformed gravitational gauge potentials (tetrad fields) $\hat{e}^a_\mu(x,\Theta)$ are obtained by contraction of the noncommutative gauge group $SO(4,1)$ to the Poincaré (inhomogeneous Lorentz) group $ISO(3,1)$. As an application, we calculate these potentials for the case of a Schwarzschild solution and define the corresponding deformed metric $\hat{g}_{\mu\nu}(x,\Theta)$. It is for the first time when such a deformed metric is given for a 4-dimensional noncommutative space–time. The corrections appear only in the second order of the expansion in Θ, i.e., there are no terms of the first order in Θ. We will give also an evaluation of the noncommutativity corrections to the red shift test of general relativity, which turns out to be extremely small for the case of the Sun.

The calculations are very tedious, so that we have used an analytical program conceived for the GRTensor II package of the Maple platform. Specific routines have been written and adapted for Maple.

Section 2 is devoted to the commutative gauge theory of the de Sitter group $SO(4,1)$ formulated on a 4-dimensional Minkowski space–time endowed with a spherical metric. Section 3 contains the results regarding the noncommutative theory. The deformed gauge potentials (tetrad fields) are obtained up to the second order of the expansion in Θ. Based on these results, we define a deformed real metric and calculate its components in the case of a Schwarzschild solution. Using the results we determine in Section 4 the deformed Schwarzschild metric. The corrections are obtained up to the second order of the noncommutativity parameters $\Theta^{\mu\nu}$. An evaluation of the value for the correction to the red shift test of general relativity is also given. Some concluding remarks and further directions of investigation are given in Section 5.

2. Commutative gauge theory

We review first the gauge theory of the de Sitter group $SO(4,1)$ on a commutative 4-dimensional Minkowski space–time endowed with the spherically symmetric metric [7]:

$$ds^2 = dr^2 + r^2\big(d\theta^2 + \sin^2\theta\, d\varphi^2\big) - c^2\,dt^2. \tag{2.1}$$

This means that the coordinates on this space–time are chosen as $(x^\mu) = (r,\theta,\varphi,ct)$, $\mu = 1,2,3,0$. The $SO(4,1)$ group is 10-dimensional and its infinitesimal generators are denoted by $M_{AB} = -M_{BA}$, $A,B = 1,2,3,0,5$. If we introduce the indices $a,b,\ldots = 1,2,3,0$, i.e., we put $A = a,5$, $B = b,5$, etc., then the generators M_{AB} can be identified with translations $P_a = M_{a5}$ and Lorentz rotations $M_{ab} = -M_{ba}$. The corresponding non-deformed gauge potentials will be denoted by $\omega^{AB}_\mu(x) = -\omega^{BA}_\mu(x)$. They are identified with the spin connection, $\omega^{ab}_\mu(x) = -\omega^{ba}_\mu(x)$, and the tetrad fields, $\omega^{a5}_\mu(x) = ke^a_\mu(x)$, where k is the contraction parameter. For the limit $k \to 0$ we obtain the $ISO(3,1)$ gauge group, i.e., the commutative Poincaré gauge theory of gravitation. The strength field associated with $\omega^{AB}_\mu(x)$ is [7]:

$$F^{AB}_\mu = \partial_\mu\omega^{AB}_\nu - \partial_\nu\omega^{AB}_\mu + \big(\omega^{AC}_\mu\omega^{DB}_\nu - \omega^{AC}_\nu\omega^{DB}_\mu\big)\eta_{CD}, \tag{2.2}$$

where $\eta_{AB} = \mathrm{diag}(1,1,1,-1,1)$. Then, we have:

$$F^{a5}_{\mu\nu} \equiv kT^a_{\mu\nu} = k\big[\partial_\mu e^a_\nu - \partial_\nu e^a_\mu + \big(\omega^{ab}_\mu e^c_\nu - \omega^{ab}_\nu e^c_\mu\big)\eta_{bc}\big], \tag{2.3}$$

$$F^{ab}_{\mu\nu} \equiv R^{ab}_{\mu\nu} = \partial_\mu\omega^{ab}_\nu - \partial_\nu\omega^{ab}_\mu + \big(\omega^{ac}_\mu\omega^{db}_\nu - \omega^{ac}_\nu\omega^{db}_\mu\big)\eta_{cd} + k\big(e^a_\mu e^b_\nu - e^a_\nu e^b_\mu\big), \tag{2.4}$$

where $\eta_{ab} = \mathrm{diag}(1,1,1,-1)$. The Poincaré gauge theory that we are using has the geometric structure of the Riemann–Cartan space $U(4)$ with curvature and torsion [6]. The quantity $T^a_{\mu\nu}$ is interpreted as the torsion tensor and $R^{ab}_{\mu\nu}$ as the curvature tensor of the Riemann–Cartan space–time defined by the gravitational fields $e^a_\mu(x)$ and $\omega^{ab}_\mu(x)$. By imposing the condition of null torsion $T^a_{\mu\nu} = 0$, one can solve for $\omega^{ab}_\mu(x)$ in terms of $e^a_\mu(x)$, i.e., the spin connection components are determined by tetrads (they are not independent fields).

Now, we consider a particular form of spherically gauge fields of the $SO(4,1)$ group given by the following ansatz [7]:

$$e^1_\mu = \left(\frac{1}{A},0,0,0\right), \qquad e^2_\mu = (0,r,0,0), \qquad e^3_\mu = (0,0,r\sin\theta,0), \qquad e^0_\mu = (0,0,0,A), \tag{2.5}$$

$$\begin{aligned}
&\omega^{12}_\mu = (0,W,0,0), \qquad \omega^{13}_\mu = (0,0,Z\sin\theta,0), \qquad \omega^{23}_\mu = (0,0,-\cos\theta,V),\\
&\omega^{10}_\mu = (0,0,0,U), \qquad \omega^{20}_\mu = \omega^{30}_\mu = (0,0,0,0),
\end{aligned} \tag{2.6}$$

where A, U, V, W and Z are functions only of the three-dimensional radius. The non-zero components of $T^a_{\mu\nu}$ and $R^{ab}_{\mu\nu}$ were obtained in [7] using an analytical program designed for GRTensor II package of Maple:

$$T^0_{01} = -\frac{AA' + U}{A}, \qquad T^2_{03} = rV\sin\theta T^2_{12} = \frac{A + W}{A}, \qquad T^3_{02} = -rV, \qquad T^3_{13} = \frac{(A + Z)\sin\theta}{A}, \tag{2.7}$$

and respectively

$$\begin{aligned} &R^{01}_{01} = U', \qquad R^{23}_{01} = -V', \qquad R^{13}_{23} = (Z - W)\cos\theta, \qquad R^{01}_{01} = -UW, \qquad R^{13}_{01} = -VW, \qquad R^{03}_{03} = -UZ\sin\theta, \\ &R^{12}_{03} = VZ\sin\theta R^{12}_{12} = W', \qquad R^{23}_{23} = (1 - ZW)\sin\theta, \qquad R^{13}_{13} = Z'\sin\theta, \end{aligned} \tag{2.8}$$

where A', U', V', W' and Z' denote the derivatives of first order with respect to the r-coordinate.

If we use (2.7), then the condition of null-torsion $T^a_{\mu\nu} = 0$ gives the following constraints:

$$U = -AA', \qquad V = 0, \qquad W = Z = -A, \tag{2.9}$$

as we have already mentioned. Then, from the field equations for $e^a_\mu(x)$

$$R^a_\mu - \frac{1}{2}Re^a_\mu = 0, \tag{2.10}$$

where $R^a_\mu = R^{ab}_{\mu\nu}\bar{e}^\nu_b$, $R = R^{ab}_{\mu\nu}\bar{e}^\mu_a\bar{e}^\nu_b$ and $\bar{e}^\mu_a$ is the inverse of e^a_μ, we obtain the solution [7]

$$A^2 = 1 - \frac{\alpha}{r}, \tag{2.11}$$

where α is an arbitrary constant of integration. For $\alpha = \frac{2GM}{c^2}$ we obtain the commutative Schwarzschild solution (G is the Newton constant and M is the mass of the point-like source of the gravitational field). The corresponding metric

$$g_{\mu\nu} = \eta_{ab}e^a_\mu e^b_\nu \tag{2.12}$$

has the following non-zero components

$$g_{11} = \left(1 - \frac{2GM}{c^2 r}\right)^{-1}, \qquad g_{22} = \frac{g_{33}}{\sin\theta} = r, \qquad g_{00} = -\left(1 - \frac{2GM}{c^2 r}\right). \tag{2.13}$$

We emphasize that this solution is obtained from the commutative $SO(4,1)$ gauge theory with a contraction $k \to 0$ to the Poincaré group $ISO(3,1)$.

We will follow now Ref. [8] in order to obtain a deformation of gravitation by gauging the noncommutative de Sitter $SO(4,1)$ group. Then, by contraction to the Poincaré (inhomogeneous Lorentz) group $ISO(3,1)$ we will obtain the deformed gauge fields $\hat{e}^a_\mu(x, \Theta)$. In the next two sections we will calculate these fields for the case of the Schwarzschild solution and define the corresponding deformed metric $\hat{g}_{\mu\nu}(x,\Theta)$ up to the second order of the expansion in Θ.

3. Deformed gauge fields

We assume that the noncommutative structure of the space–time is determined by the condition

$$[x^\mu, x^\nu] = i\Theta^{\mu\nu}, \tag{3.1}$$

where $\Theta^{\mu\nu} = -\Theta^{\nu\mu}$ are constant (canonical) parameters. To develop the noncommutative gauge theory, we introduce the star product "*" between the functions f and g defined over this space–time:

$$(f * g)(x) = f(x)e^{\frac{i}{2}\Theta^{\mu\nu}\overleftarrow{\partial_\mu}\overrightarrow{\partial_\nu}}g(x). \tag{3.2}$$

The gauge fields for the noncommutative case are denoted by $\hat{\omega}^{AB}_\mu(x,\Theta)$, and they are subject to the reality conditions [8,10,11]:

$$\hat{\omega}^{AB+}_\mu(x,\Theta) = -\hat{\omega}^{BA}_\mu(x,\Theta), \qquad \hat{\omega}^{AB}_\mu(x,\Theta)^r \equiv \hat{\omega}^{AB}_\mu(x,-\Theta) = -\hat{\omega}^{BA}_\mu(x,\Theta), \tag{3.3}$$

with "+" denoting the complex conjugate.

By expanding $\hat{\omega}^{ab}_\mu(x,\Theta)$ in powers of the noncommutative parameter Θ,

$$\hat{\omega}^{AB}_\mu(x,\Theta) = \omega^{AB}_\mu(x) - i\Theta^{\nu\rho}\omega^{AB}_{\mu\nu\rho}(x) + \Theta^{\nu\rho}\Theta^{\lambda\tau}\omega^{AB}_{\mu\nu\rho\lambda\tau}(x) + \cdots, \tag{3.4}$$

the constraints (3.3) imply the properties

$$\omega^{AB}_\mu(x) = -\omega^{BA}_\mu(x), \qquad \omega^{AB}_{\mu\nu\rho}(x) = \omega^{BA}_{\mu\nu\rho}(x), \qquad \omega^{AB}_{\mu\nu\rho\lambda\tau}(x) = -\omega^{BA}_{\mu\nu\rho\lambda\tau}(x), \quad \ldots. \tag{3.5}$$

Using the Seiberg–Witten map [2], one obtains the following noncommutative corrections up to the second order [8]:

$$\omega^{AB}_{\mu\nu\rho}(x) = \frac{1}{4}\{\omega_\nu, \partial_\rho\omega_\mu + R_{\rho\mu}\}^{AB}, \tag{3.6}$$

$$\begin{aligned}\omega^{AB}_{\mu\nu\rho\lambda\tau}(x) = \frac{1}{32}\big(&-\big\{\omega_\lambda, \partial_\tau\{\omega_\nu, \partial_\rho\omega_\mu + R_{\rho\mu}\}\big\} + 2\big\{\omega_\lambda, \{R_{\tau\nu}, R_{\mu\rho}\}\big\} - \big\{\omega_\lambda, \{\omega_\nu, D_\rho R_{\tau\mu} + \partial_\rho R_{\tau\mu}\}\big\} \\ &- \big\{\{\omega_\nu, \partial_\rho\omega_\lambda + R_{\rho\lambda}\}, (\partial_\tau\omega_\mu + R_{\tau\mu})\big\} + 2\big[\partial_\nu\omega_\lambda, \partial_\rho(\partial_\tau\omega_\mu + R_{\tau\mu})\big]\big)^{AB},\end{aligned} \tag{3.7}$$

where

$$\{\alpha,\beta\}^{AB} = \alpha^{AC}\beta^B_C + \beta^{AC}\alpha^B_C, \qquad [\alpha,\beta]^{AB} = \alpha^{AC}\beta^B_C - \beta^{AC}\alpha^B_C \tag{3.8}$$

and

$$D_\mu R^{AB}_{\rho\sigma} = \partial_\mu R^{AB}_{\rho\sigma} + \big(\omega^{AC}_\mu R^{DB}_{\rho\sigma} + \omega^{BC}_\mu R^{DA}_{\rho\sigma}\big)\eta_{CD}. \tag{3.9}$$

As in the commutative case, we write $\hat{\omega}^{a5}_\mu(x,\Theta) = k\hat{e}^a_\mu(x,\Theta)$ and $\hat{\omega}^{55}_\mu(x,\Theta) = k\phi_\mu(x,\Theta)$. Then we impose the condition of null torsion $T^a_{\mu\nu} = 0$ and not $\hat{T}^a_{\mu\nu} = 0$, since by contraction $k \to 0$ the quantity $\phi_\mu(x,\Theta)$ will drop out [8]. Using (3.6) and (3.7) we obtain, in the limit $k \to 0$, the deformed tetrad fields $\hat{e}^a_\mu(x,\Theta)$ up to the second order:

$$\hat{e}^a_\mu(x,\Theta) = e^a_\mu(x) - i\Theta^{\nu\rho}e^a_{\mu\nu\rho}(x) + \Theta^{\nu\rho}\Theta^{\lambda\tau}e^a_{\mu\nu\rho\lambda\tau}(x) + O\big(\Theta^3\big), \tag{3.10}$$

where

$$e^a_{\mu\nu\rho} = \frac{1}{4}\big[\omega^{ac}_\nu\partial_\rho e^d_\mu + \big(\partial_\rho\omega^{ac}_\mu + R^{ac}_{\rho\mu}\big)e^d_\nu\big]\eta_{cd}, \tag{3.11}$$

$$\begin{aligned}e^a_{\mu\nu\rho\lambda\tau} = \frac{1}{32}\big[&2\{R_{\tau\nu}, R_{\mu\rho}\}^{ab}e^c_\lambda - \omega^{ab}_\lambda\big(D_\rho R^{cd}_{\tau\mu} + \partial_\rho R^{cd}_{\tau\mu}\big)e^m_\nu\eta_{dm} - \big\{\omega_\nu, (D_\rho R_{\tau\mu} + \partial_\rho R_{\tau\mu})\big\}^{ab}e^c_\lambda \\ &- \partial_\tau\big\{\omega_\nu, (\partial_\rho\omega_\mu + R_{\rho\mu})\big\}^{ab}e^c_\lambda - \omega^{ab}_\lambda\partial_\tau\big(\omega^{cd}_\nu\partial_\rho e^m_\mu + \big(\partial_\rho\omega^{cd}_\mu + R^{cd}_{\rho\mu}\big)e^m_\nu\big)\eta_{dm} + 2\partial_\nu\omega^{ab}_\lambda\partial_\rho\partial_\tau e^c_\mu \\ &- 2\partial_\rho\big(\partial_\tau\omega^{ab}_\mu + R^{ab}_{\tau\mu}\big)\partial_\nu e^c_\lambda - \big\{\omega_\nu, (\partial_\rho\omega_\lambda + R_{\rho\lambda})\big\}^{ab}\partial_\tau e^c_\mu \\ &- \big(\partial_\tau\omega^{ab}_\mu + R^{ab}_{\tau\mu}\big)\big(\omega^{cd}_\nu\partial_\rho e^m_\lambda + \big(\partial_\rho\omega^{cd}_\lambda + R^{cd}_{\rho\lambda}\big)e^m_\nu\eta_{dm}\big)\big]\eta_{bc}.\end{aligned} \tag{3.12}$$

We define also the complex conjugate $\hat{e}^{a+}_\mu(x,\Theta)$ of the deformed tetrad fields given in (3.10) by:

$$\hat{e}^{a+}_\mu(x,\Theta) = e^a_\mu(x) + i\Theta^{\nu\rho}e^a_{\mu\nu\rho}(x) + \Theta^{\nu\rho}\Theta^{\lambda\tau}e^a_{\mu\nu\rho\lambda\tau}(x) + O\big(\Theta^3\big). \tag{3.13}$$

Then we can introduce a deformed metric by the formula:

$$\hat{g}_{\mu\nu}(x,\Theta) = \frac{1}{2}\eta_{ab}\big(\hat{e}^a_\mu * \hat{e}^{b+}_\nu + \hat{e}^b_\mu * \hat{e}^{a+}_\nu\big). \tag{3.14}$$

We can see that this metric is, by definition, a real one, even if the deformed tetrad fields $\hat{e}^a_\mu(x,\Theta)$ are complex quantities.

4. Second order corrections to Schwarzschild solution

Using the ansatz (2.5)–(2.6), we can determine the deformed Schwarzschild metric. To this end, we have to obtain first the corresponding components of the tetrad fields $\hat{e}^a_\mu(x,\Theta)$ and their complex conjugated $\hat{e}^{a+}_\mu(x,\Theta)$ given by Eqs. (3.10) and (3.13). With the definition (3.14) it is possible then to obtain the components of the deformed metric $\hat{g}_{\mu\nu}(x,\Theta)$.

To simplify the calculations, we choose the coordinate system so that the parameters $\Theta^{\mu\nu}$ are given as:

$$\Theta^{\mu\nu} = \begin{pmatrix} 0 & \Theta & 0 & 0 \\ -\Theta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mu,\nu = 1,2,3,0. \tag{4.1}$$

The constant quantity Θ, which determines the noncommutativity of the space–time coordinates, has the dimension L^2 (square of length).

The non-zero components of the tetrad fields $\hat{e}^a_\mu(x,\Theta)$ are:

$$\hat{e}^1_1 = \frac{1}{A} + \frac{A''}{8}\Theta^2 + O\big(\Theta^3\big), \tag{4.2}$$

$$\hat{e}^1_2 = -\frac{i}{4}(A + 2rA')\Theta + O\big(\Theta^3\big), \tag{4.3}$$

$$\hat{e}_2^2 = r + \frac{1}{32}\big(7AA' + 12rA'^2 + 12rAA''\big)\Theta^2 + O\big(\Theta^3\big), \tag{4.4}$$

$$\hat{e}_3^3 = r\sin\theta - \frac{i}{4}(\cos\theta)\Theta + \frac{1}{8}\left(2rA'^2 + rAA'' + 2AA' - \frac{A'}{A}\right)(\sin\theta)\Theta^2 + O\big(\Theta^3\big), \tag{4.5}$$

$$\hat{e}_0^0 = A + \frac{1}{8}\big(2rA'^3 + 5rAA'A'' + rA^2A''' + 2AA'^2 + A^2A''\big)\Theta^2 + O\big(\Theta^3\big), \tag{4.6}$$

where A', A'', A''' are first, second and third derivatives of $A(r)$, respectively. The complex conjugated components can be easily obtained from these expressions.

Then, using the definition (3.14), we obtain the following non-zero components of the deformed metric $\hat{g}_{\mu\nu}(x,\Theta)$ up to the second order:

$$\begin{aligned}
\hat{g}_{11}(x,\Theta) &= \frac{1}{A^2} + \frac{1}{4}\frac{A''}{A}\Theta^2 + O\big(\Theta^4\big),\\
\hat{g}_{22}(x,\Theta) &= r^2 + \frac{1}{16}\big(A^2 + 11rAA' + 16r^2A'^2 + 12r^2AA''\big)\Theta^2 + O\big(\Theta^4\big),\\
\hat{g}_{33}(x,\Theta) &= r^2\sin^2\theta + \frac{1}{16}\left[4\left(2rAA' - r\frac{A'}{A} + r^2AA'' + 2r^2A'^2\right)\sin^2\theta + \cos^2\theta\right]\Theta^2 + O\big(\Theta^4\big),\\
\hat{g}_{00}(x,\Theta) &= -A^2 - \frac{1}{4}\big(2rAA'^3 + rA^3A''' + A^3A'' + 2A^2A'^2 + 5rA^2A'A''\big)\Theta^2 + O\big(\Theta^4\big).
\end{aligned} \tag{4.7}$$

For $\Theta \to 0$ we obtain the commutative Schwarzschild solution with $A^2 = 1 - \frac{\alpha}{r}$ (see Eq. (2.11)).

It is interesting to remark that, if we choose the parameters $\Theta^{\mu\nu}$ as in (4.1), then the deformed metric $\hat{g}_{\mu\nu}(x,\Theta)$ is diagonal as it is in the commutative case. But, in general, for arbitrary $\Theta^{\mu\nu}$, the deformed metric $\hat{g}_{\mu\nu}(x,\Theta)$ is not diagonal even if the commutative (non-deformed) one has this property. Therefore, we can conclude that the noncommutativity modifies the structure of the gravitational field.

For the Schwarzschild solution we have:

$$A(r) = \sqrt{1 - \frac{\alpha}{r}}, \quad \alpha = \frac{2GM}{c^2}. \tag{4.8}$$

The function $A(r)$ is dimensionless, but its derivatives A', A'' and A''' have respectively the dimensions L^{-1}, L^{-2} and L^{-3}. As a consequence, all the components of the deformed metric $\hat{g}_{\mu\nu}(x,\Theta)$ in (4.7) have the correct dimensions.

Now, if we introduce (4.8) into (4.7), then we obtain the deformed Schwarzschild metric. Its non-zero components are:

$$\begin{aligned}
\hat{g}_{11} &= \frac{1}{1-\frac{\alpha}{r}} - \frac{\alpha(4r-3\alpha)}{16r^2(r-\alpha)^2}\Theta^2 + O\big(\Theta^4\big),\\
\hat{g}_{22} &= r^2 + \frac{2r^2 - 17\alpha r + 17\alpha^2}{32r(r-\alpha)}\Theta^2 + O\big(\Theta^4\big),\\
\hat{g}_{33} &= r^2\sin^2\theta + \frac{(r^2 + \alpha r - \alpha^2)\cos^2\theta - \alpha(2r-\alpha)}{16r(r-\alpha)}\Theta^2 + O\big(\Theta^4\big),\\
\hat{g}_{00} &= -\left(1 - \frac{\alpha}{r}\right) - \frac{\alpha(8r - 11\alpha)}{16r^4}\Theta^2 + O\big(\Theta^4\big).
\end{aligned} \tag{4.9}$$

As we see from (4.9), in the limit $\theta = 0$, the usual Schwarzschild solution is obtained, as it should be, due to the use of perturbation in θ according to the Seiberg–Witten map approach. We also note that all the non-zero components of the metric in (4.9), with the exception of $\hat{g}_{00}$, acquire a singularity in the θ^2-correction term at the value $r = \alpha$. This singularity pertains also to all the invariants of the theory, such as the Ricci scalar, etc. For details and also on the cosmological consequences, we refer the reader to [20].

We can evaluate then the contributions of these corrections to the tests of general relativity. For example, if we consider the red shift of the light propagating in a gravitational field [21], then we obtain for the case of the Sun:

$$\frac{\Delta\lambda}{\lambda} = \frac{\alpha}{2R} - \frac{\alpha(8R - 11\alpha)}{32R^4}\Theta^2 + O\big(\Theta^4\big), \tag{4.10}$$

where R is the radius of the Sun. Since for the Sun we have $\alpha = \frac{2GM}{c^2} = 2.95 \times 10^3$ m and $R = 6.955 \times 10^8$ m, then we obtain from (4.10):

$$\frac{\Delta\lambda}{\lambda} = 2 \times 10^{-6} - 2.19 \times 10^{-24}\ \mathrm{m}^{-2}\ \Theta^2 + O\big(\Theta^4\big). \tag{4.11}$$

The noncommutativity correction has a value which is much too small, compared to the value which results from general relativity, and the precision of the measurement is not sufficient to put a reasonable bound on the noncommutativity parameter.

5. Concluding remarks

Using the Seiberg–Witten map we have determined the noncommutativity corrections to the Schwarzschild solution up to the second order in the parameters $\Theta^{\mu\nu}$. Following Ref. [7], we reviewed first a de Sitter gauge theory of gravitation over a spherical symmetric commutative Minkowski space–time. Then, a deformation of the gravitational field has been constructed along Ref. [8] by gauging the noncommutative de Sitter $SO(4,1)$ group and using Seiberg–Witten map. The corresponding space–time is also of Minkowski type but endowed now with spherical noncommutative coordinates. We determined the deformed gauge fields up to the second order in the noncommutativity parameters $\Theta^{\mu\nu}$. The deformed gravitational gauge potentials (tetrad fields) $\hat{e}^a_\mu(x,\Theta)$ have been obtained by contraction of the noncommutative gauge group $SO(4,1)$ to the Poincaré (inhomogeneous Lorentz) group $ISO(3,1)$. As an application, we have calculated these potentials for the case of the Schwarzschild solution and defined the corresponding deformed metric $\hat{g}_{\mu\nu}(x,\Theta)$. The corrections appear only in the second order of the expansion in Θ, i.e., there are no first order correction terms. For the calculations we used an analytical program conceived for the GRTensor II package of the Maple platform.

We have considered also the red shift test in the noncommutative theory and determined the value of the relative displacement $\frac{\Delta\lambda}{\lambda}$ for the case of Sun. The result shows that the correction is too small to have observable effects.

Having found the Schwarzschild solution for a noncommutative theory of gravity we have been breaking new ground towards approaching the black-hole physics on noncommutative space–time [20].

Acknowledgements

We are grateful to J. Gracia-Bondia, A. Kobakhidze and M.R. Setare for useful discussions and suggestions. A.T. acknowledges the Project No. 121720 of the Academy of Finland. G.Z. acknowledges the support by the CNCSIS-UEFISCSU grant 620 of the Minister of Education, Research and Youth of Romania.

References

[1] S. Doplicher, K. Fredenhagen, J.E. Roberts, Commun. Math. Phys. 172 (1995) 187.
[2] N. Seiberg, E. Witten, JHEP 9909 (1999) 032, hep-th/9908142.
[3] R. Utiyama, Phys. Rev. 101 (1956) 1597.
[4] T.W.B. Kibble, J. Math. Phys. 2 (1961) 212.
[5] T. Eguchi, P.B. Gilkey, A.J. Hanson, Phys. Rep. 66 (1980) 213;
F.W. Hehl, J.D. McCrea, E.W. Mielke, Y. Ne'eman, Phys. Rep. 258 (1995) 1, gr-qc/9402012;
F. Gronwald, F.W. Hehl, On the gauge aspects of gravity, in: Erice 1995, Quantum Gravity, pp. 148–198, gr-qc/9602013;
A.H. Chamseddine, Int. J. Geom. Meth. Mod. Phys. 3 (2006) 149, hep-th/0511074 (special issue in honour of R. Utiyama).
[6] F.W. Hehl, P. von der Heyde, D. Kerlick, J. Neste, Rev. Mod. Phys. 48 (1976) 393;
Y. Choquet-Bruhat, C. deWitt-Morette, M. Dillard-Bleick, Analysis, Manifolds and Physics, North-Holland, Amsterdam, 1977;
M. Blagojević, Gravitation and Gauge Symmetries, Institute of Physics Publishing, Bristol, 2002.
[7] G. Zet, V. Manta, S. Babeti, Int. J. Mod. Phys. C 14 (2003) 41.
[8] A.H. Chamseddine, Phys. Lett. B 504 (2001) 33, hep-th/0009153.
[9] J.W. Moffat, Phys. Lett. B 491 (2000) 345, hep-th/0007181;
A.H. Chamseddine, Commun. Math. Phys. 218 (2001) 283, hep-th/0005222;
A.H. Chamseddine, J. Math. Phys. 44 (2003) 2534, hep-th/0202137;
S. Carlip, Rep. Prog. Phys. 64 (2001) 885, gr-qc/0108040;
M.A. Cardella, D. Zanon, Class. Quantum Grav. 20 (2003) L95, hep-th/0212071;
S.I. Vacaru, J. Math. Phys. 46 (2005) 042503, gr-qc/0307103;
P. Mukherjee, A. Saha, Phys. Rev. D 74 (2006) 027702, hep-th/0605287.
[10] L. Bonora, M. Schnabl, M. Sheikh-Jabbari, A. Tomasiello, Nucl. Phys. B 589 (2000) 461, hep-th/0006091.
[11] B. Jurco, S. Schraml, P. Schupp, J. Wess, Eur. Phys. J. C 17 (2000) 521, hep-th/0006246.
[12] M. Chaichian, P.P. Kulish, K. Nishijima, A. Tureanu, Phys. Lett. B 604 (2004) 000, hep-th/0408069;
M. Chaichian, P. Prešnajder, A. Tureanu, Phys. Rev. Lett. 94 (2005) 151602, hep-th/0409096.
[13] P. Aschieri, C. Blohmann, M. Dimitrijevic, F. Meyer, P. Schupp, J. Wess, Class. Quantum Grav. 22 (2005) 3511, hep-th/0504183.
[14] L. Álvarez-Gaumé, F. Meyer, M.A. Vazquez-Mozo, Nucl. Phys. B 75 (2006) 392, hep-th/0605113.
[15] X. Calmet, A. Kobakhidze, Phys. Rev. D 72 (2005) 045010, hep-th/0506157;
X. Calmet, A. Kobakhidze, Phys. Rev. D 74 (2006) 047702, hep-th/0605275.
[16] A. Kobakhidze, hep-th/0603132.
[17] M. Chaichian, A. Tureanu, Phys. Lett. B 637 (2006) 199, hep-th/0604025.
[18] M. Chaichian, A. Tureanu, G. Zet, Phys. Lett. B 651 (2007) 319, hep-th/0607179.
[19] M. Chaichian, A. Tureanu, R.B. Zhang, X. Zhang, hep-th/0612128.
[20] M. Chaichian, M.R. Setare, A. Tureanu, G. Zet, arXiv: 0711.4546 [hep-th].
[21] S. Weinberg, Gravitation and Cosmology, John Wiley and Sons, New York, 1972.