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Schwarzschild-de-Sitter Solution in Quantum Gauge Theory of Gravity

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Abstract *We use the theory based on the gravitational gauge group G to obtain a spherical symmetric solution of the field equations for the gravitational potentials on a Minkowski space-time. The gauge group G is defined and then we introduce the gauge-covariant derivative D_μ. The strength tensor of the gravitational gauge field is also obtained and a gauge-invariant Lagrangian including the cosmological constant is constructed. A model whose gravitational gauge potentials $A_\mu^\alpha(x)$ have spherical symmetry, depending only on the radial coordinate r is considered and an analytical solution of these equations, which induces the Schwarzschild-de-Sitter metric on the gauge group space, is then determined. All the calculations have been performed by GR Tensor II computer algebra package, running on the Maple V platform, along with several routines that we have written for our model.*

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1 Introduction

In the gauge theory of gravity, the Poincaré group is localized usually, but the gravitational field is not characterized by gauge potentials. It is represented by metric field and the effects of gravity are described by the curvature of the space-time. The gauge theory based on the Poincaré group proves to be correct at the classical level but it is non-renormalizable. However, the gauge theories are fundamental in the field theory and, in particular, in the elementary particle physics.[1] The three non-gravitational interactions (electromagnetic, weak, and strong) are completely described by means of gauge theory in the framework of the Standard Model (SM). First of all, the gauge theory of the unitary groups SU(N) is of fundamental importance in elementary particle physics. The SM of strong and electroweak interactions is based on the gauge theory of SU(3) $\times$ SU(2) $\times$ U(1) group. In addition, the "Grand Unification" is described by the gauging of SU(5) group.[1] Secondly, the Poincaré group (Lorentz transformations and space-time translations) is also of a fundamental importance in any field theory. After pioneering works of Utiyama,[2] Sciama,[3,4] and Kibble[5] it was recognized that gravitation also can be formulated as a gauge theory. The gauge groups considered in gauge theory of gravitation are Poincaré group,[6] de-Sitter group,[7] affine group,[8] etc. It is believed that the formulation of gravity as a gauge theory on a Minkowski space-time could lead to a consistent quantum theory of gravity.[9,10]

Recently, Wu[10] proposed a gauge theory of General Relativity (GR) based on the gravitational gauge group (G). In his theory, the gravitational interaction is considered as a fundamental interaction in a flat Minkowski space-time, and not as space-time geometry. The gravitation gauge group G consists of generalized space-time translations, and the gravity is described by gauge potentials. Contrarily, if there is gravitational field in space-time, the space-time metric will not be equivalent to Minkowski metric, and space-time will become curved. In other words, in the traditional gravitational gauge theory, the gravity is formulated in curved space-time. In this paper, we will not follow this way. The underlying point of view of this new quantum gauge general relativity, developed by Wu, is that the gravity is treated as a kind of physical interactions in flat space-time and the gravitational field is represented by gauge potential. For this reason, we will not introduce the concept of curved space-time to study quantum gravity in this paper. So, the space-time is always flat, the gravitational field is represented by gauge potential, and gravitational interactions are always treated as physical interactions.

In this paper we use the theory based on the gravitational gauge group G to obtain a spherical symmetric solution of the field equations for the gravitational potentials on a Minkowski space-time. In Sec. 2 we define the gravitational gauge group G and then we introduce the gauge-covariant derivative D_μ. The strength tensor of the gravitational gauge field is obtained and a gauge invariant Lagrangian including the cosmological constant is constructed starting from that proposed by Wu. The field equations of the gauge potentials are written with a gravitational energy-momentum tensor $(T_g)_{\mu\nu}$ on the right-hand side. This tensor has the same expression as in Ning Wu's theory.

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Section 3 is devoted to the case of a model whose gravitational gauge potentials $A_\mu^\alpha(x)$ have spherical symmetry, depending only on the radial coordinate r. The corresponding non-null components of the strength tensor $F_{\mu\nu}$ of the gravitational gauge field are obtained and then the gauge field equations are written. In Sec. 4, an analytical solution of these equations, which induce the Schwarzschild-de-Sitter metric on the gauge group space, is then determined.

All the calculations from Secs. 3 and 4 have been performed by GR Tensor II computer algebra package, running on the *Maple* V platform, along with several routines that we have written for our model. The integration of the field equation was also performed by this computer algebra package. The program is described in Sec. 5, where we list also the instructions which allows to define and calculate the quantities and the equations of the model.

2 Gravitational Gauge Group and Field Equations

Wu proposed a new quantum gauge theory of general relativity based on the gravitational gauge group G as local symmetry.[1−5] In this theory, the gravitation is treated as a physical interaction in a Minkowski (flat) space-time M and the gravitational field is represented by gauge potentials whose definition follows. It differs on General Relativity (GR) where the gravitational field is described by the metric tensor of a curved space-time within a geometric model.

In this section we resume the theory of Wu and extend the results to the case when the cosmological constant Λ is introduced in model. In the following sections we suppose that the gravitational gauge potentials have spherical symmetry and obtain a Schwarzschild-de-Sitter type solution of the field equations.

The infinitesimal transformations of the group G are, as usually, of the form,[10]

$$U(\varepsilon) \cong 1 - \varepsilon^\alpha P_\alpha\,, \quad \alpha = 1,2,3,0\,, \tag{1}$$

where ε^α are the infinitesimal parameters of the group and $P_\alpha = -\mathrm{i}\partial_\alpha$ are the generators of gauge group. It is known that these generators commute each other,

$$[P_\alpha, P_\beta] = 0\,. \tag{2}$$

However, in accord with Wu model, this does not mean that the group G is Abelian, that is, its elements do not commute,[10]

$$[U(\varepsilon_1), U(\varepsilon_2)] \neq 0\,. \tag{3}$$

It is emphasized that there is a difference between the group T of space-time translations and the gravitational gauge group G. Space-time translations of T are coordinate (passive) transformations, that is, the objects or fields (physical system) are fixed in space-time, while the coordinates themselves undergo transformations. Contrarily, under the transformations of G, the space-time system of coordinates is fixed and the physical system undergoes (active) transformations.

As usually, one introduces the gauge gravitational field with values into the Lie algebra of the group G:

$$A_\mu(x) = A_\mu^\alpha(x) P_\alpha\,, \quad \mu = 1,2,3,0\,, \tag{4}$$

where $A_\mu^\alpha(x)$ are the gravitational gauge potentials, and then a gauge-covariant derivative is defined as

$$D_\mu = \partial_\mu - \mathrm{i} g A_\mu(x)\,. \tag{5}$$

Here g denotes the gauge coupling constant of the gravitational interactions. The corresponding strength tensor $F_{\mu\nu}(x) = F_{\mu\nu}^\alpha(x) P_\alpha$, with values in the Lie algebra of G, has the components[10]

$$F_{\mu\nu}^\alpha(x) = G_\mu^\beta \partial_\beta A_\nu^\alpha - G_\nu^\beta \partial_\beta A_\mu^\alpha\,, \tag{6}$$

where

$$G_\mu^\alpha(x) = \delta_\mu^\alpha - g A_\mu^\alpha(x) \tag{7}$$

are new gauge potentials. We suppose that these new potentials admit the inverses $\bar{G}_\alpha^\mu(x)$ with the usual properties:

$$\bar{G}_\alpha^\mu G_\mu^\beta = \delta_\alpha^\beta\,, \quad \bar{G}_\alpha^\mu G_\nu^\alpha = \delta_\nu^\mu\,. \tag{8}$$

Following Refs. [10] ∼ [12], we define a metric tensor on the gravitational gauge group space by:

$$g_{\alpha\beta} = \eta_{\mu\nu} \bar{G}_\alpha^\mu \bar{G}_\beta^\nu\,, \tag{9a}$$

$$g^{\alpha\beta} = \eta^{\mu\nu} G_\mu^\alpha G_\nu^\beta\,, \tag{9b}$$

where $\eta_{\mu\nu} = \mathrm{diag}\,(1,1,1,-1)$ is the metric tensor of the Minkowski space-time M, and $\eta^{\mu\nu}$ denotes its inverse.

In order to generalize the results of Wu to the case when the cosmological constant is present in the model,[13,14] we consider the integral of action for the gravitational gauge potentials under the form:

$$S = \int \sqrt{-\det(g_{\alpha\beta})}\, L\, \mathrm{d}^4 x\,, \tag{10}$$

where $\det(g_{\alpha\beta})$ is the determinant of the metric tensor $g_{\alpha\beta}$, and L is the Lagrangian density of the gravitational field,

$$L_0 = -\frac{1}{16}\eta^{\mu\rho}\eta^{\nu\sigma} g_{\alpha\beta} F_{\mu\nu}^\alpha F_{\rho\sigma}^\beta - \frac{1}{8}\eta^{\mu\rho} \bar{G}_\beta^\nu \bar{G}_\alpha^\sigma F_{\mu\nu}^\alpha F_{\rho\sigma}^\beta + \frac{1}{4}\eta^{\mu\rho} \bar{G}_\alpha^\nu \bar{G}_\beta^\sigma F_{\mu\nu}^\alpha F_{\rho\sigma}^\beta + \frac{\Lambda}{2g^2}\,. \tag{11}$$

Taking $\delta S = 0$ with respect to gravitational gauge potentials $A_\mu^\alpha(x)$,[15,16] we obtain the following field equations:

$$\partial_\mu \Big(\frac{1}{4}\eta^{\mu\rho}\eta^{\nu\sigma} g_{\alpha\beta} F_{\rho\sigma}^\sigma - \frac{1}{4}\eta^{\nu\rho} F_{\rho\alpha}^\mu + \frac{1}{4}\eta^{\mu\rho} F_{\rho\alpha}^\nu - \frac{1}{2}\eta^{\mu\rho}\delta_\alpha^\nu F_{\rho\beta}^\beta + \frac{1}{2}\eta^{\nu\rho}\delta_\alpha^\mu F_{\rho\beta}^\beta \Big) - \frac{\Lambda}{2g}\bar{G}_\alpha^\nu = -g (T_g)_\alpha^\nu\,, \tag{12}$$

where $(T_g)_\alpha^\nu$ is the gravitational energy-momentum tensor considered as the source of the gravitational field.[10] This tensor has the same expression as in Wu's works[10−12] and will not be given here. Its expression is given in the computing program listed in Sec. 5. In what follows, we will write the field equations (12) for the case when the gauge potentials have spherical symmetry,[17−19] and obtain a solution of Schwarzschild-de-Sitter type.

3 Spherically Symmetric Model

We consider now a model of gauge theory of gravitation having the gravitational gauge group G as local group of symmetry. The base manifold is a four-dimensional Minkowski space-time M endowed with spherical coordinates $(x^1, x^2, x^3, x^0) = (r, \theta, \varphi, t)$. The gravitational gauge potentials $A^\alpha_\mu(x)$ are chosen under the form:

$$A^\alpha_\mu = \begin{pmatrix} A(r) & 0 & 0 & 0 \\ 0 & \dfrac{r-1}{gr} & 0 & 0 \\ 0 & 0 & \dfrac{r\sin\theta - 1}{gr\sin\theta} & 0 \\ 0 & 0 & 0 & -\dfrac{A(r)}{1-gA(r)} \end{pmatrix}, \tag{13}$$

where $A(r)$ is a function of the radial coordinate r only, and g is the coupling constant of the gravitational field introduced into the previous section. Then, the new gauge potentials $G^\alpha_\mu(x)$, defined by Eq. (7), have the form:

$$G^\alpha_\mu = \begin{pmatrix} 1-gA(r) & 0 & 0 & 0 \\ 0 & \dfrac{1}{r} & 0 & 0 \\ 0 & 0 & \dfrac{1}{r\sin\theta} & 0 \\ 0 & 0 & 0 & \dfrac{1}{1-gA(r)} \end{pmatrix}. \tag{14}$$

Their inverse components are

$$\bar{G}^\mu_\alpha = \begin{pmatrix} \dfrac{1}{1-gA(r)} & 0 & 0 & 0 \\ 0 & r & 0 & 0 \\ 0 & 0 & r\sin\theta & 0 \\ 0 & 0 & 0 & 1-gA(r) \end{pmatrix}. \tag{15}$$

Having these potentials chosen, we can calculate the components $F^\alpha_{\mu\nu}(x)$ of the corresponding strength tensor given in Eq. (6). For such a purpose we used an analytical computing program written by us in GR Tensor II package working on the *Maple* platform. The description of this program is given in Sec. 5. Here we only list the non-null components of the strength tensor:

$$F^2_{12} = \sin\theta\,, \quad F^3_{13} = \frac{1-gA(r)}{gr^2}\,,$$
$$F^0_{10} = -\frac{A'(r)}{1-gA(r)}\,, \quad F^3_{23} = \frac{\cos\theta}{gr^2\sin^2\theta}\,. \tag{16}$$

We calculate also the components $g_{\alpha\beta}$ of the metric tensor and its inverse which are defined in Eqs. (9a) and (9b). The results are

$$g_{\alpha\beta} = \begin{pmatrix} \frac{1}{(1-gA(r))^2} & 0 & 0 & 0 \\ 0 & r^2 & 0 & 0 \\ 0 & 0 & r^2\sin^2\theta & 0 \\ 0 & 0 & 0 & -(1-gA(r))^2 \end{pmatrix}, \tag{17}$$

and respectively

$$g^{\alpha\beta} = \begin{pmatrix} (1-gA(r))^2 & 0 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 & 0 \\ 0 & 0 & \frac{1}{r^2\sin\theta} & 0 \\ 0 & 0 & 0 & -\frac{1}{(1-gA(r))^2} \end{pmatrix}. \tag{18}$$

Now, we list here only a few of the simpler components of the gravitational energy-momentum tensor:

$$T^1_1 = -\frac{A(r)(2-gA(r))}{2gr^2(1-gA(r))} + \frac{A'(r)}{gr} - \frac{1}{g^2r^2\sin^3\theta} + \frac{1}{2g^2r^2\sin\theta}\,, \tag{19a}$$

$$T^2_1 = \frac{\cos\theta}{g^2r^3\sin^2\theta}\,, \quad T^1_2 = -\frac{(1-gA(r))}{2g^2r^2\sin^2\theta}\,, \tag{19b}$$

where $A'(r)$ denotes the derivative of the function $A(r)$ with respect to the r variable. The other non-null components which are not listed here are T^2_2, T^3_3, and T^0_0, and they can be easily obtained by running the computing program described in Sec. 5.

4 Solution of Field Equations

We can write now the field equations (12) of the spherically symmetric gauge gravitational potentials $A^\alpha_\mu(x)$ chosen in previous Section. Using the analytical computing program described in Sec. 5, we obtained the following four field equations:

$$\frac{1}{2gr^2(1-gA(r))}(2g^2rA(r)A'(r) - 2grA'(r) + g^2A(r)^2 - 2gA(r) + \Lambda r^2) = 0\,, \tag{20}$$

$$\frac{1}{2g}\big(2g^2A(r)A'(r) + g^2rA(r)A''(r) - 2gA'(r) + g^2rA'(r)^2 - grA''(r) + \Lambda r\big) = 0\,, \tag{21}$$

$$\frac{\sin\theta}{2g}\big(2g^2A(r)A'(r) + g^2rA(r)A''(r) - 2gA'(r) + g^2rA'(r)^2 - grA''(r) + \Lambda r\big) = 0\,, \tag{22}$$

$$\frac{1-gA(r)}{2gr^2}\big(2g^2rA(r)A'(r) - 2grA'(r) + g^2A(r)^2 - 2gA(r) + \Lambda r^2\big) = 0\,. \tag{23}$$

where $A''(r)$ is the derivative of second order of the function $A(r)$ with respect to variable r. It is easy to verify that these equations are equivalent if $1 - gA(r) \neq 0$, so that we have only one independent field equations for one unknown function $A(r)$. The constraint $1 - gA(r) \neq 0$ is in accord with the definition (13) of the gauge potentials $A^\alpha_\mu(x)$.

In order to obtain a solution for $A(r)$, we consider the field equation (20) only, and introduce a new unknown function,

$$y(r) = (1 - gA(r))^2 \,. \tag{24}$$

Then, equation (20) becomes

$$ry'(r) + y(r) = 1 - \Lambda r^2 \,, \tag{25}$$

or, equivalently,

$$(ry(r))' = 1 - \Lambda r^2 \,. \tag{26}$$

The integration of Eq. (26) is directly and we finally obtain the solution,

$$y(r) = 1 + \frac{\alpha}{r} - \frac{\Lambda}{3} r^2 \,, \tag{27}$$

where α is an arbitrary constant of integration. Then, introducing Eq. (27) into Eq. (24), we obtain the solution

$$A(r) = \frac{1 \pm \sqrt{1 + (\alpha/r) - (\Lambda r^2/3)}}{g} \,. \tag{28}$$

If we choose $\alpha = -2Gm$, where m is supposed to be the mass of the point-like source of our gravitational field, then the result (28) corresponds to a Schwarzschild-de-Sitter type solution having the square of the line element [see Eq. (17)]:

$$\mathrm{d}s^2 = \frac{\mathrm{d}r^2}{1 - (2Gm/r) - (\Lambda r^2/3)} + r^2(\mathrm{d}\theta^2 + \sin^2\theta \, \mathrm{d}\varphi^2) - \Big(1 - \frac{2Gm}{r} - \frac{\Lambda}{3} r^2\Big) \mathrm{d}t^2 \,. \tag{29}$$

We remember that this line element is defined on the gravitational gauge group space and that the space-time remains a Minkowski (flat) one.

5 Analytical Computing Program

In order to calculate the components of indexed objects (in particular tensors) or defining new tensors by using GR Tensor II package, first of all, we have to specify the space-time geometry.[17,20] In our computing program we load the metric of the space-time using the qload (mink 2) command. The Minkowski metric of the space-time is denoted in our program by *eta*1 {*miu niu*} and its inverse by *eta*1*inv* {*miu niu*}.

The command grdef( ) is included to facilitate the specification of new tensors in a simple and natural manner. It allows tensors to be defined either as an equation in terms of previously defined tensors, or by manual entry of their components. Inner and outer products of tensors, symmetrization, and derivatives can all be specified as part of the tensor definitions. Furthermore, index symmetries of the newly defined tensors can be included. The integration of the field equations has been done by command "with (DEtools, odeadvisor)".

The gauge potentials $A^\alpha_\mu(x)$, denoted by $A\{\wedge alpha\ miu\}$, were introduced by manual entry of their components, and the new gauge potentials $G^\alpha_\mu(x)$, defined in Eq. (7), have been denoted by $Gb\,\{\wedge alpha\ miu\}$ and their inverses by $Gbinv\,\{\wedge alpha\ miu\}$.

The analytical program allows to calculate: the components of the strength tensor field $F^\alpha_{\mu\nu}$, denoted by $F\,\{\wedge alpha\ miu\ niu\}$, the components of the metric $g_{\alpha\beta}$, denoted by $gb\,\{alpha\ beta\}$, the components $g^{\alpha\beta}$ of its inverse, denoted by $gbinv\{\wedge alpha\ \wedge beta\}$ of the gravitational energy-momentum tensor $(T_g)^\nu_\alpha$, denoted by $Tg\,\{\wedge niu\ alpha\}$, and the field equations, denoted in our program by $EQ\,\{\wedge niu\ alpha\}$. The expression under the derivative ∂_μ on the left-hand side of the gauge field equations (12) has been denoted by $EX\,\{\wedge miu\ \wedge niu\ alpha\}$. The differential equation (25) or (26) has been denoted by *ode* and then it was integrated by the commands: *odeadvisor (ode)* and *dsolve (ode)*. The comments are inserted as instructions noted by symbol #.

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