Available online at www.sciencedirect.com
ScienceDirect
Physics Letters B 651 (2007) 319–323
PHYSICS LETTERS B
www.elsevier.com/locate/physletb

Twist as a symmetry principle and the noncommutative gauge theory formulation

M. Chaichian [a], A. Tureanu [a,*], G. Zet [a,b]

[a] High Energy Physics Division, Department of Physical Sciences, University of Helsinki, Helsinki Institute of Physics, PO Box 64, FIN-00014 Helsinki, Finland
[b] Department of Physics, "Gh. Asachi" Technical University, Bd. D. Mangeron 67, 700050 Iasi, Romania

Received 15 December 2006; received in revised form 2 May 2007; accepted 12 June 2007

Available online 14 June 2007

Editor: L. Alvarez-Gaumé

Abstract

Based on the analysis of the most natural and general ansatz, we conclude that the concept of twist symmetry, originally obtained for the noncommutative space–time, cannot be extended to include internal gauge symmetry. The case is reminiscent of the Coleman–Mandula theorem. Invoking the supersymmetry may reverse the situation.

1. Introduction

For field theories on the noncommutative space–time with Heisenberg-like commutation relation

$$[\hat{x}_\mu, \hat{x}_\nu] = i\theta_{\mu\nu}, \tag{1.1}$$

where $\theta_{\mu\nu}$ is an antisymmetric matrix, the traditional framework has been the Weyl–Moyal correspondence, by which to each field operator $\Phi(\hat{x})$ corresponds a Weyl symbol $\Phi(x)$, defined on the commutative counterpart of the space–time. An essential aspect of this correspondence is that, in the action functional, the products of field operators, e.g. $\Phi(\hat{x})\Psi(\hat{x})$ is replaced by the Moyal $\star$-product of Weyl symbols, $\Phi(x) \star \Psi(x)$, where

$$\star = \exp\left(\frac{i}{2}\theta_{\mu\nu} \overleftarrow{\partial}^\mu \overrightarrow{\partial}^\nu\right). \tag{1.2}$$

In this correspondence, the operator commutation relation (1.1) becomes

$$[x_\mu, x_\nu]_\star = x_\mu \star x_\nu - x_\nu \star x_\mu = i\theta_{\mu\nu} \tag{1.3}$$

and noncommutative models have been built by simply taking their commutative counterparts and replacing the usual multiplication by $\star$-product (see [1] and references therein).

It turns out that such noncommutative models, although they lack Lorentz symmetry, are invariant under the twisted Poincaré algebra [2], deformed with the Abelian twist element

$$\mathcal{F} = \exp\left(\frac{i}{2}\theta^{\mu\nu} P_\mu \otimes P_\nu\right), \tag{1.4}$$

where P_μ are the generators of translations. The twist induces on the algebra of representation of the Poincaré algebra the deformed multiplication

$$\begin{aligned} m \circ (\phi \otimes \psi) = \phi\psi \to m_\star \circ (\phi \otimes \psi) \\ = m \circ \mathcal{F}^{-1}(\phi \otimes \psi) \equiv \phi \star \psi, \end{aligned} \tag{1.5}$$

which is nothing else but the $\star$-product (1.2). Important consequences for the representation theory of the noncommutative fields arise from here.

In parallel with the NC QFT models, NC gauge theories have been constructed using the same prescription, of taking the Lagrangian of the commutative theory and replacing the usual multiplication by the $\star$-product (1.2) [3]. By construction, such theories are twisted-Poincaré invariant, if we use the twist element (1.4). However, as far as the gauge invariance is concerned, the models are invariant under $\star$-gauge transformations. For example, in the case of the gauge $U_\star(n)$ group, an

* Corresponding author.
E-mail address: anca.tureanu@helsinki.fi (A. Tureanu).

0370-2693/$ – see front matter
doi:10.1016/j.physletb.2007.06.026

arbitrary element of the group will be

$$U(x) = \exp_\star\big(i\alpha^a(x)T_a\big), \tag{1.6}$$

where T_a, $a = 1, \dots, n^2$ are the generators of the $U(n)$ group, with the algebra $[T_a, T_b] = if_{abc}T_c$, $\alpha^a(x)$, $a = 1, \dots, n^2$ are the gauge parameters and the $\star$-exponential means

$$\exp_\star\big(i\alpha^a(x)T_a\big) = 1 + i\alpha^a(x)T_a + \frac{1}{2!}(i)^2\alpha^a(x) \star \alpha^b(x)T_aT_b + \cdots. \tag{1.7}$$

The use of the $\star$-product in the formulation of gauge theories imposes strict constraints on the noncommutative gauge symmetry, among which is the fact that only NC gauge $U(n)$ groups close (and not $SU(n)$). Moreover, there is a no-go theorem [4] stating that only certain representations of the gauge group are allowed (fundamental, anti-fundamental and adjoint) (see also [5]) and the matter fields can be charged under at most two gauge groups. We have to emphasize that although these gauge theories are twisted-Poincaré invariant, the $\star$-gauge transformations are implemented separately, in the sense that the coproduct of the gauge generators is not twisted with the Abelian twist (1.4).

Recently, an attempt was made to twist also the gauge algebra, i.e. to extend the Poincaré algebra by the gauge algebra, as semidirect product, and to twist the coproduct of the gauge generators with the same Abelian twist (1.4) [6,7]. The result seemed to be spectacular: the same theories, which previously were shown to be subject to the no-go theorem [4,5], were now claimed to be invariant under any usual (not noncommutative) gauge group and to admit any representations, just as in the commutative case. The latter approach was shown [8] however to be in conflict with the very idea of gauge transformations, since it assumed implicitly that if a field is transformed according to a given representation of the gauge algebra, then its derivatives of any order also transform according to the representations of the gauge algebra, which is obviously not the case.

The question arises whether the concept of *twist* appears as a *symmetry principle* in constructing NC field theories: any symmetry that such theories may enjoy, be it space–time or internal symmetry, global or local, should be formulated as a twisted symmetry. In pursuit of this idea, in this Letter we take the most general ansatz for a non-Abelian twist, which, in the absence of the gauge interaction, reduces to the Abelian twist (1.4). We shall show that the twisting of the gauge transformations is not possible, in a manner compatible with the representations of the gauge algebra and keeping at the same time the Moyal space defined by (1.3) as underlying space of the theory.

2. Necessity of a symmetry principle for noncommutative field theories

The necessity of a new approach to noncommutative gauge theories arises both from internal gauge symmetries and the gravitational theory.

Noncommutative internal gauge symmetry

The essential physical implication of the twisted Poincaré symmetry is that the representation content of this quantum symmetry and of usual Poincaré symmetry are the same. As a consequence, the asymptotic fields are the same in commutative and noncommutative field theories. This legitimates the perturbative approach to NC QFT, starting from the representation content of Poincaré algebra (for details, see [2]).

On the other hand, any application such as model building has to circumvent one way or another the no-go theorem [4,5]. The ways for by-passing the restrictions imposed by the no-go theorem (e.g. by dressing the fields with Wilson lines or by invoking enveloping algebra-valued fields) are not unique and lack justification. A twisted symmetry principle would provide a truly solid base for the formulation of noncommutative gauge theories.

Noncommutative gravitational theory

NC gravitational effects have been recently calculated [9] from string theory with antisymmetric background field, i.e. in the same theory as the one which gave rise in the low-energy limit to the usual noncommutative field theories [1]. It turns out that, in the case of NC gravitational interactions, string theory contains a much richer dynamics than the one of the theories constructed [10] in terms of Moyal $\star$-products alone, by twisting the algebra of diffeomorphisms with the frame-dependent twist element (1.4). The inconsistencies are caused by the fact that the deformation of general coordinate transformations is not so far done in a frame-independent manner. In other words, when the twist element is chosen as (1.4), the frame-dependent Moyal $\star$-product is fixed once for all by the choice of the twist and thus it does not transform at all. Since the diffeomorphisms are basically external gauge transformations, the situation is technically similar [8] to the one which results when one attempted to deform the internal gauge transformations with the same twist element (1.4).

It thus appears that the currently studied noncommutative gauge and gravitational theories show incompatibilities with respect to the twisted Poincaré symmetry, besides internal inconsistencies mentioned above. It is therefore desirable to find a general symmetry principle (and the applicability of the twisted Poincaré symmetry leads to the conclusion that this general symmetry will be a quantum one), starting from which one could construct noncommutative gauge and gravitational theories free of internal contradictions.

3. Gauge transformations and the concept of twist

Let us consider the Lie algebra $\mathcal{G}$ as an internal symmetry. The infinitesimal generators of the algebra $\mathcal{G}$ are denoted by T_a, $a = 1, \dots, m$, and they satisfy the commutation relations

$$[T_a, T_b] = if_{abc}T_c. \tag{3.1}$$

Subsequently we gauge the algebra of internal symmetry $\mathcal{G}$ and define

$$\alpha(x) = \alpha^a(x)T_a \tag{3.2}$$

as Hermitian generators of the infinitesimal gauge transformations. Since the gauge generators do not commute with the generators of the global Poincaré algebra, we can extend the Poincaré algebra $\mathcal{P}$ by semidirect product with the gauge generators, the purpose being to eventually deform the enveloping algebra of this semidirect product, $\mathcal{U}(\mathcal{P} \ltimes \mathcal{G})$, considered as a Hopf algebra, by an appropriately chosen twist [11] (see also [12]). The algebra of representation for $\mathcal{U}(\mathcal{P} \ltimes \mathcal{G})$ is the algebra of fields $\mathcal{A}$, defined on the Minkowski space. The action of the generators of the Hopf algebra on the fields is the usual one, even upon twisting. In particular, for the infinitesimal gauge transformations we have

$$\delta_\alpha \Phi(x) = i\alpha(x)\Phi(x), \qquad \delta_\alpha \Phi^\dagger(x) = -i\Phi^\dagger(x)\alpha(x), \tag{3.3}$$

where $\alpha(x)$ is defined in (3.2) and $\Phi(x) \in \mathcal{A}$. We emphasize the absence of a star-product in (3.3), unlike the case of the traditional noncommutative gauge theories [3].

The principle of gauge invariance [13] requires the introduction of gauge fields if we want the action of a theory to be symmetric under local transformations. By their transformation properties, the gauge fields have the role to compensate for the terms arising from the fact that the derivatives of fields (in the kinetic terms) do not transform according to the representations of the *gauge* algebra, like the fields themselves do. With the gauge fields $A_\mu(x) = A_\mu^a(x)T_a$ transforming in the adjoint representation of the gauge algebra as

$$\delta_\alpha A_\mu(x) = i\big[\alpha(x), A_\mu(x)\big] + \partial_\mu \alpha(x), \tag{3.4}$$

one constructs the covariant derivative

$$D_\mu = \partial_\mu - iA_\mu, \tag{3.5}$$

such that the combination $D_\mu \Phi(x)$ transforms again like the field itself under gauge transformations, i.e.

$$\delta_\alpha D_\mu \Phi(x) = i\alpha(x)\big(D_\mu \Phi(x)\big). \tag{3.6}$$

Moreover, applying any number of covariant derivatives to a field, the result will transform in the same way:

$$\delta_\alpha D_{\mu_1} \cdots D_{\mu_n} \Phi(x) = i\alpha(x)\big(D_{\mu_1} \cdots D_{\mu_n} \Phi(x)\big), \tag{3.7}$$

in other words,

$$\delta_\alpha D_{\mu_1} \cdots D_{\mu_n} = \big[\alpha(x), D_{\mu_1} \cdots D_{\mu_n}\big]. \tag{3.8}$$

We have to point out that, even upon twisting $\mathcal{U}(\mathcal{P} \ltimes \mathcal{G})$, the covariant derivative has to act as usual on the matter fields, i.e. without any star-product between the gauge field $A_\mu(x)$ and the matter field $\Phi(x)$. This is because the covariant derivative is in effect a linear combination of generators of $\mathcal{U}(\mathcal{P} \ltimes \mathcal{G})$:

$$D_\mu = i\big(P_\mu - A_\mu^a T_a\big), \tag{3.9}$$

where the realization of P_μ on the Minkowski space, $P_\mu = -i\partial_\mu$, is used.

4. Non-Abelian twist of $\mathcal{U}(\mathcal{P} \ltimes \mathcal{G})$

In [8] it was shown in detail that the use of the Abelian twist (1.4) for deforming the Hopf algebra $\mathcal{U}(\mathcal{P} \ltimes \mathcal{G})$ is not compatible with the concept of gauge transformations. We recall that the reason for this conflict is the fact that the derivatives of a field do not transform according to the representations of the gauge algebra, as the fields themselves do.

However, the covariant derivatives of a field transform exactly according to the same representation as the field itself, as we have mentioned above. Thus the option of defining a *non-Abelian twist element* involving covariant derivatives naturally occurs:

$$\mathcal{T} = \exp\left(-\frac{i}{2}\theta^{\mu\nu} D_\mu \otimes D_\nu + \mathcal{O}\big(\theta^2\big)\right), \tag{4.1}$$

where the terms of higher order in θ contain as well products of covariant derivatives and remain to be found.[1] The twist element (4.1) has to satisfy the twist conditions [12], i.e.:

$$\mathcal{T}_{12}(\Delta_0 \otimes id)\mathcal{T} = \mathcal{T}_{23}(id \otimes \Delta_0)\mathcal{T}, \tag{4.2}$$

$$(\epsilon \otimes id)\mathcal{T} = 1 = (id \otimes \epsilon)\mathcal{T}, \tag{4.3}$$

where Δ_0 is the symmetric coproduct of the generators of the Lie algebra $\mathcal{P} \ltimes \mathcal{G}$, such that

$$\Delta_0(Y) = Y \otimes 1 + 1 \otimes Y, \quad \text{for } Y \in \mathcal{P} \ltimes \mathcal{G}, \tag{4.4}$$

$\epsilon : \mathcal{U}(\mathcal{P} \ltimes \mathcal{G}) \to C$ is the counit, satisfying

$$(id \otimes \epsilon) \circ \Delta_0 = id = (\epsilon \otimes id) \circ \Delta_0 \tag{4.5}$$

and $\mathcal{T}_{12} = \mathcal{T} \otimes 1$ and $\mathcal{T}_{23} = 1 \otimes \mathcal{T}$. By the twist element (4.1) one deforms the symmetric coproduct (4.4):

$$\Delta_0(Y) \mapsto \mathcal{T}\Delta_0(Y)\mathcal{T}^{-1}. \tag{4.6}$$

The twisting of the coproduct of the generators requires a corresponding deformation of the product of fields into a star product, which we shall denote by $\bigstar$, to differentiate it from the Weyl–Moyal $\star$-product:

$$\begin{aligned} m \circ (\Phi \otimes \Psi) = \Phi\Psi \to m_\bigstar \circ (\Phi \otimes \Psi) &= m \circ \mathcal{T}^{-1}(\Phi \otimes \Psi) \\ &\equiv \Phi \bigstar \Psi. \end{aligned} \tag{4.7}$$

Remark that the actual form of the covariant derivatives in the second term of (4.7) is given by the respective fields on which they act, i.e. by their representation under the gauge algebra. The associativity of the $\bigstar$-product corresponding to the non-Abelian twist is equivalent to the fulfillment of the twist condition (4.2).

Since the purpose of the non-Abelian twist (4.1) is to generalize the Abelian twist (1.4), in a manner which would consistently include the noncommutative gauge transformations, the new star-product induced by the non-Abelian twist has to

[1] For the relaxation of the exponential form (4.1) to an arbitrary invertible function for the twist element $\mathcal{T}$, see the end of this section. The exponential form (4.1) however is taken, to start with, by requiring a "correspondence principle", that the twist (4.1) would reduce to the Abelian one (1.4) in the absence of gauge fields.

reduce to the usual Weyl–Moyal star-product for ordinary functions. Indeed, ordinary functions on the Minkowski space have to be considered in the 1-dimensional (trivial) representation of the gauge group G, i.e.

$$e^{i\alpha^a(x)T_a} f(x) = f(x) + i\alpha^a(x)T_a f(x) + \cdots = f(x),$$

which implies $T_a f(x) = 0$. This means that for ordinary functions we have $D_\mu f(x) = \partial_\mu f(x)$, from which it should follow:

$$\begin{aligned} &m \circ \mathcal{T}^{-1}\big(f(x)\otimes g(x)\big) \\ &\quad = m \circ \exp\left(\frac{i}{2}\theta^{\mu\nu}\partial_\mu \otimes \partial_\nu\right)\big(f(x)\otimes g(x)\big) \\ &\quad \equiv f(x) \star g(x). \end{aligned} \tag{4.8}$$

The same result has to apply to the fields in the trivial (1-dimensional) representation of the gauge group. It is then clear, by taking $f(x) = x_\mu$ and $g(x) = x_\nu$ in the above, that the non-Abelian twist would lead to gauge theories on the same noncommutative space–time with the commutation relation (1.3). Therefore, in finding the concrete form of the non-Abelian twist (4.1) we have to fulfill the constraint that the exponential has to reduce to the usual exponential function of (1.4) when its argument contains usual commuting derivatives.

If we take in the non-Abelian twist (4.1) only the term of first order in θ, one can straightforwardly show that the twist condition (4.2) is not fulfilled already in the second order in θ, while (4.3) and (4.8) are. The second order terms which do not cancel in (4.2) are, in the l.h.s.

$$\begin{aligned} &\frac{1}{2}\left(-\frac{i}{2}\right)^2 \theta^{\mu\nu}\theta^{\rho\sigma}(D_\rho \otimes D_\mu \otimes D_\sigma D_\nu + D_\mu \otimes D_\rho \otimes D_\sigma D_\nu \\ &\quad + 2D_\mu D_\rho \otimes D_\nu \otimes D_\sigma + 2D_\mu \otimes D_\nu D_\rho \otimes D_\sigma) \end{aligned} \tag{4.9}$$

and in the r.h.s.

$$\begin{aligned} &\frac{1}{2}\left(-\frac{i}{2}\right)^2 \theta^{\mu\nu}\theta^{\rho\sigma}(2D_\rho \otimes D_\mu \otimes D_\nu D_\sigma + D_\rho D_\mu \otimes D_\sigma \otimes D_\nu \\ &\quad + D_\rho D_\mu \otimes D_\nu \otimes D_\sigma + 2D_\rho \otimes D_\mu D_\sigma \otimes D_\nu). \end{aligned} \tag{4.10}$$

One may argue that there are still first order terms which were not taken into account, i.e. $\theta^{\mu\nu} 1 \otimes F_{\mu\nu}$ and $\theta^{\mu\nu} F_{\mu\nu} \otimes 1$. However, such terms will not contribute to the cancelation of (4.9) and (4.10), because they will introduce only terms in which the indices of the second rank tensor, be it on the first, second or last place, correspond to the same θ, i.e. $F_{\mu\nu} \otimes D_\rho \otimes D_\sigma$, $D_\rho \otimes F_{\mu\nu} \otimes D_\sigma$ or $D_\rho \otimes D_\sigma \otimes F_{\mu\nu}$, while the indices of the second rank tensor in the terms to be canceled of (4.9) and (4.10) correspond to different θs. Moreover, if one writes an action with the new $\bigstar$-product replacing the usual multiplication in the Lagrangian, the terms coming from $\theta^{\mu\nu} 1 \otimes F_{\mu\nu}$ and $\theta^{\mu\nu} F_{\mu\nu} \otimes 1$ will give the same contribution, upon partial integration, like the terms coming from $\theta^{\mu\nu} D_\mu \otimes D_\nu$. For this reasons we decide to omit other terms of the first order in θ except $\theta^{\mu\nu} D_\mu \otimes D_\nu$.

The second order terms not canceled in the twist condition suggest the exponent in the form of a series in θ. The general form of such a series would be cumbersome to write down, however, we can easily write the most general second order term which satisfies (4.8) and impose the twist condition (4.2) up to second order in θ.

Possible typical second order terms are:

$$\begin{aligned} &\theta^{\mu\nu}\theta^{\rho\sigma}(1 \otimes D_\mu D_\nu D_\rho D_\sigma) \quad \text{and} \\ &\quad \theta^{\mu\nu}\theta^{\rho\sigma}(D_\mu D_\nu D_\rho D_\sigma \otimes 1), \end{aligned} \tag{4.11}$$

$$\begin{aligned} &\theta^{\mu\nu}\theta^{\rho\sigma}(D_\mu \otimes D_\nu D_\rho D_\sigma) \quad \text{and} \\ &\quad \theta^{\mu\nu}\theta^{\rho\sigma}(D_\mu D_\nu D_\rho \otimes D_\sigma), \end{aligned} \tag{4.12}$$

$$\theta^{\mu\nu}\theta^{\rho\sigma}(D_\mu D_\nu \otimes D_\rho D_\sigma), \tag{4.13}$$

with all the permutations of indices of the covariant derivatives. The terms of the type (4.11) satisfy (4.8), but their structure is such, that they cannot cancel the terms which appear in the second order from the first term of the exponential. The terms of the type (4.13) do not satisfy in general (4.8) but the terms which satisfy the latter condition cannot help in the cancelation. The only terms which satisfy (4.8) and could contribute to the cancelation are (4.12) and we shall add only such terms.

Since the order of the indices is important in the terms (4.12) with permutations, there are altogether $2\frac{4!}{(4-3)!} = 2 \times 24$ of this type. However, due to the antisymmetry of the indices (μ, ν) and (ρ, σ), only 2×3 combinations of indices are independent. Thus, the most general form of (4.1) with meaningful terms of second order in θ, which satisfy (4.8), is:

$$\begin{aligned} \mathcal{T} = \exp\bigg\{ &\bigg(-\frac{i}{2}\theta^{\mu\nu} D_\mu \otimes D_\nu \\ &+ \frac{1}{2}\left(-\frac{i}{2}\right)^2 \theta^{\mu\nu}\theta^{\rho\sigma}\big[a D_\mu \otimes D_\sigma D_\nu D_\rho \\ &+ b D_\mu \otimes D_\nu D_\sigma D_\rho + c D_\mu \otimes D_\sigma D_\rho D_\nu \\ &+ a' D_\sigma D_\nu D_\rho \otimes D_\mu + b' D_\nu D_\sigma D_\rho \otimes D_\mu \\ &+ c' D_\sigma D_\rho D_\nu \otimes D_\mu\big] + \mathcal{O}\big(\theta^3\big)\bigg\}, \end{aligned} \tag{4.14}$$

where a, b, c, a', b', c' are constants which have to be determined by imposing (4.2) up to the second order in θ. Typically, the terms which do not cancel out in the twist condition are of the form $\theta\theta D \otimes D \otimes DD$, $\theta\theta DD \otimes D \otimes D$ and $\theta\theta D \otimes DD \otimes D$. Imposing the cancelation of the terms of the type $\theta\theta D \otimes D \otimes DD$, one obtains $a = -1$ and $a + b + c = 0$, while from the terms of the type $\theta\theta DD \otimes D \otimes D$ one obtains $a' = -1$ and $a' + b' + c' = 0$. However, when requiring the cancelation of the terms of the type $\theta\theta D \otimes DD \otimes D$, one obtains $a + a' = 2$ and $a + b + c = -(a' + b' + c')$. Obviously the three conditions cannot be satisfied simultaneously, consequently there are no second order terms, formulated in terms of covariant derivatives, which can lead to the fulfillment of the twist condition (4.2) up to the second order in θ.

Omitting the requirement that the non-Abelian twist should reduce to the usual twist (i.e. with the usual Moyal $\star$-product) when gauge fields are absent allows for other possible second order terms, such as (4.13), with all possible permutation of the indices of covariant derivatives. We have verified that even by admitting such terms, the twist condition (4.2) cannot be satisfied. We have also verified that, by relaxing the requirement of exponential form for the twist as in (4.1) to an arbitrary invertible function $F(X)$, i.e. by taking the first and second derivatives $F'(0)$ and $F''(0)$ (the coefficients of the θ-expansion of the twist) to be arbitrary, the twist condition (4.2) still cannot be satisfied. Thus the result is general and is not based on the requirement of "correspondence principle".

We can therefore conclude that a non-Abelian twist element, which would generalize (1.4) in a gauge covariant manner cannot exist.

5. Conclusions

In this Letter we have tackled the question whether the twist could be regarded as a symmetry principle for the NC field and gauge theories. To this end, we proposed a new, non-Abelian, twist element (4.1) for the formulation of noncommutative gauge theories on Moyal spaces. The new star-product arising in this way, containing covariant derivatives in place of the usual derivatives, would insure both the twisted Poincaré and the twisted gauge invariance of noncommutative (gauge) field theories. We have shown, however, that the non-Abelian twist element, although gauge covariant, does not satisfy the twist conditions. The result does not depend on the exponential form for the twist as in (4.1), but is valid for an arbitrary invertible functional form. Having in view also the analysis of [8], which showed that the Abelian twist (1.4) cannot be used for twisting gauge transformations, it appears that there is no way to reconcile the twist condition and the gauge invariance principle. Let us mention that by using the Seiberg–Witten map [1], which provides a connection between a NC gauge symmetry and the corresponding commutative one as a power series in the non-commutativity parameter $\theta_{\mu\nu}$, the resulting Lagrangian or action [14] cannot be brought to the form given by a twist.

It is intriguing that the *external* Poincaré symmetry and the *internal* gauge symmetry cannot be unified under a common twist. The situation is reminiscent of the Coleman–Mandula theorem [15] (for a pedagogical presentation and other references, see [16]), although not entirely, since the Coleman–Mandula theorem concerns global symmetry and simple groups. However, one can envisage that supersymmetry [17], due to its intrinsic internal symmetry, may reverse the situation, and a noncommutative supersymmetric gauge theory can be constructed by means of a twist [18].

Acknowledgements

We are indebted to M. Hayakawa, P.P. Kulish, K. Nishijima, P. Prešnajder, M. Sheikh-Jabbari and A.A. Zheltukhin for useful discussions.

References

[1] N. Seiberg, E. Witten, JHEP 9909 (1999) 032, hep-th/9908142.
[2] M. Chaichian, P. Kulish, K. Nishijima, A. Tureanu, Phys. Lett. B 604 (2004) 98, hep-th/0408069;
M. Chaichian, P. Prešnajder, A. Tureanu, Phys. Rev. Lett. 94 (2005) 151602, hep-th/0409096.
[3] M. Hayakawa, hep-th/9912167;
M. Hayakawa, Phys. Lett. B 478 (2000) 394, hep-th/9912094.
[4] M. Chaichian, P. Prešnajder, M.M. Sheikh-Jabbari, A. Tureanu, Phys. Lett. B 526 (2002) 132, hep-th/0107037.
[5] S. Terashima, Phys. Lett. B 482 (2000) 276, hep-th/0002119.
[6] D.V. Vassilevich, Mod. Phys. Lett. A 21 (2006) 1279, hep-th/0602185.
[7] P. Aschieri, M. Dimitrievic, F. Meyer, S. Schraml, J. Wess, Lett. Math. Phys. 78 (2006) 61, hep-th/0603024.
[8] M. Chaichian, A. Tureanu, Phys. Lett. B 637 (2006) 199, hep-th/0604025.
[9] L. Álvarez-Gaumé, F. Meyer, M. Vázquez-Mozo, Nucl. Phys. B 753 (2006) 92, hep-th/0605113.
[10] P. Aschieri, C. Blohmann, M. Dimitrievic, F. Meyer, P. Schupp, J. Wess, Class. Quantum Grav. 22 (2005) 3511, hep-th/0504183;
P. Aschieri, M. Dimitrievic, F. Meyer, J. Wess, Class. Quantum Grav. 23 (2006) 1883, hep-th/0510059.
[11] V.G. Drinfeld, Leningrad Math. J. 1 (1990) 321;
N.Yu. Reshetikhin, Lett. Math. Phys. 20 (1990) 331.
[12] V. Chari, A. Pressley, A Guide to Quantum Groups, Cambridge Univ. Press, Cambridge, 1994;
M. Chaichian, A. Demichev, Introduction to Quantum Groups, World Scientific, Singapore, 1996.
[13] C.N. Yang, R.L. Mills, Phys. Rev. 96 (1954) 191;
R. Utiyama, Phys. Rev. 101 (1956) 1597.
[14] J. Madore, S. Schraml, P. Schupp, J. Wess, Eur. Phys. J. C 16 (2000) 161, hep-th/0001203.
[15] S. Coleman, J. Mandula, Phys. Rev. 159 (1967) 1251.
[16] S. Weinberg, The Quantum Theory of Fields, Supersymmetry, vol. III, Cambridge Univ. Press, 2005.
[17] R. Haag, J.T. Lopuszanski, M. Sohnius, Nucl. Phys. B 88 (1975) 257.
[18] Work in progress.