REGULARIZATION IN QUANTUM GAUGE THEORY OF GRAVITATION WITH DE SITTER INNER SYMMETRY

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Abstract

We study the regularization of the gauge theory of gravitation using the de Sitter group as symmetry of the model. The method of generalized zeta-function is used to realize the regularization and the gauge group is considered as an internal symmetry. An effective integral of action is obtained and a comparison with other results is given.

1 INTRODUCTION

Most of the existing gauge theories of gravitation adopt a geometrical description of gravity. Namely, the Poincaré group is considered partly as a space-time partly as an internal symmetry group. The local extension of its space-time part becomes then the diffeomorphism group and the gauge theory is invariant under general coordinate transformations and local Lorentz frame rotations. Therefore, this local symmetry group is connected with the geometry of the space-time.

It is possible also to consider space-time symmetries (for example Poincaré or de Sitter in this paper) as purely inner symmetries [1, 2]. This leads to a description of the gauge theory of gravitation which is in a complete analogy with the description of inner symmetries as groups of generalized "rotations" in field space.

In this paper we consider the group de Sitter (DS) as purely inner symmetry and develop a gauge theory of gravitation. We obtain an effective integral of action which automatically includes the cosmological constant. The method of generalized zeta-function is used to study the regularization of the theory.

In Section 2 we introduce the DS gauge group and give in an explicitly form its equation of structures. The gauge covariant derivative is introduced as usually, considering the DS group as an internal symmetry and introducing the corresponding gauge fields. The strength field is defined as the commutator of two gauge covariant derivatives.

The regularization of the theory is studied in Section 3, using the method of generalized zeta function. The change of the partition function with respect to scale transform is calculated for the case of a spinor Dirac field interacting with the gravitational field described by the gauge potentials. Then, a minimal field gauge action, compatible with regularization requirements and including the cosmological constant, is determined.

Finally, some concluding remarks are given in the Section 4. It is emphasized that in our model there is no any direct interrelation between gravity and the structure of space-time. At quantum level it may conceptually be easier to deal with a field theoretical description of gravitation free of any geometrical aspects.

2 DE SITTER GAUGE THEORY

We consider a gauge theory of gravitation having de Sitter (DS) group as local symmetry. Let X_A, $A = 1, 2, \cdots, 10$, be a basis of DS Lie algebra with the corresponding equations of structure given by [1]

$$[X_A, X_B] = i f_{AB}{}^C X_C \tag{1}$$

where $f_{AB}{}^C$ are the constants of structure whose expressions will be given below [see Eq. (3)].

In order to write the constant of structures $f_{AB}{}^C$ in a compact form, we use the following notations for the index A:

$$A = \begin{cases} a & = 0, 1, 2, 3 \\ [ab] & = [01], [02], [03], [12], [13], [23] \end{cases} . \tag{2}$$

This means that A can stand for a single index like 2 as well as for a pair of indices like [01], [12], etc. The infinitesimal generators X_A are interpreted as: $X_A = P_a$ (energy-momentum operators) and $X_{[ab]} = M_{ab}$ (angular momentum operators) with the property $M_{ab} = -M_{ba}$. The constants of structure $f_{AB}{}^C$

have then the following expressions:

$$\begin{cases} f_{bc}{}^{a} & = f_{c[de]}{}^{ab} = f_{[bc][de]}{}^{a} = 0 \\ f_{cd}{}^{[ab]} & = 4\lambda^2 \left(\delta_c{}^b \delta_d{}^a - \delta_c{}^a \delta_d{}^b \right) \\ f_{b[cd]}{}^{a} & = -f_{[cd]b}{}^{a} = \frac{1}{2} \left(\eta_{bc} \delta_d{}^a - \eta_{bd} \delta_c{}^a \right) \\ f_{[ab][cd]}{}^{[ef]} & = \frac{1}{4} \left(\eta_{bc} \delta_a{}^e \delta_d{}^f - \eta_{ac} \delta_b{}^e \delta_d{}^f + \eta_{ad} \delta_b{}^e \delta_c{}^f - \eta_{bd} \delta_a{}^e \delta_c{}^f \right) - e \longleftrightarrow f \end{cases} \tag{3}$$

where λ is a real parameter, and $\eta_{ab} = diag\,(1, -1, -1, -1)$ is the Minkowski metric of the space-time. In fact, here we have a deformation of the de-Sitter Lie algebra having λ as parameter. Considering the contraction $\lambda \to 0$ we obtain the Poincaré Lie algebra, i.e., the group DS contracts to the Poincaré group.

Now we introduce the local DS gauge transformation and the corresponding gauge covariant derivative ∇_a, considering DS as an internal group of symmetry. As usually in any gauge theory, we have

$$\nabla_a = \partial_a + B_a, \tag{4}$$

together with the following decomposition of B_a with respect to the infinitesimal generators P_a and M_{ab}

$$B_a = -iB_a{}^b P_b + \frac{i}{2} B_a{}^{bc} M_{bc}. \tag{5}$$

The corresponding generators of the DS group in the field space have the expressions:

$$P_a = i\partial_a + \lambda K_a, \quad M_{ab} = i\left(x_a \partial_b - x_b \partial_a\right) + \frac{1}{2}\Sigma_{ab}, \tag{6}$$

where K_a are the "translation" de Sitter generators and Σ_{ab} the spin angular momentum operators. The last one (Σ_{ab}) satisfy commutation relations of the same form as M_{ab} and K_a have the expression [2]:

$$K_a = i\left(2\eta_{ab} x^b x^c - \sigma^2 \delta_a{}^c\right) \partial_c, \quad \sigma^2 = \eta_{ab} x^a x^b. \tag{7}$$

We can also decompose B_a with respect to ∂_a and Σ_{ab} as follows:

$$B_a = \left[B_a{}^b + \lambda B_a{}^d \left(2\eta_{dc} x^c x^b - \sigma^2 \delta_d{}^b\right)\right] \partial_b + \frac{i}{4} B_a{}^{bc} \Sigma_{bc}. \tag{8}$$

Introducing (8) into Eq. (4) and denoting

$$e_a{}^b = \delta_a{}^b + \lambda B_a{}^d \left(2\eta_{dc} x^c x^b - \sigma^2 \delta_d{}^b\right) + B_a{}^{bc} x_c, \tag{9}$$

we obtain

$$\nabla_a = e_a{}^b \partial_b + \frac{i}{4} B_a{}^{bc} \Sigma_{bc}. \tag{10}$$

Because in our model the coordinate and DS gauge transformations are strictly separated, we emphasize that the introduction of $B_a{}^b$, $B_a{}^{bc}$ and $e_a{}^b$ gauge fields has no implications on the structure of the underlying space-time, which is assumed to be (M_4, η) endowed with the Minkowski metric η.

Abbreviating

$$d_a = e_a{}^b \partial_b, \quad B_a = \frac{i}{4} B_a{}^{bc} \Sigma_{bc}, \tag{11}$$

where Σ_{ab} must be considered into the Lorentz group representation it acts on, we can write the gauge covariant derivative (10) under the simple form:

$$\nabla_a = d_a + B_a. \tag{12}$$

The derivative d_a can be just considered as a translation gauge covariant derivative [3]. In order to obtain the tensor (field strength operator) F_{ab} of the gauge fields, we introduce the non-covariant decomposition

$$[d_a, d_b] = H_{ab}{}^c d_c. \tag{13}$$

The quantity $H_{ab}{}^c$ is expressed in terms of $e_a{}^b$ as:

$$H_{ab}{}^c = \overline{e}_m{}^c \left(e_a{}^d \partial_d e_b{}^m - e_b{}^d \partial_d e_a{}^m \right), \tag{14}$$

where $\overline{e}_m{}^c$ is the matrix inverse of $e_c{}^n$, i.e. $\overline{e}_m{}^c e_c{}^n = \delta_m{}^n$. Using the definition of the field strength operator in a gauge theory, we have:

$$F_{ab} = [\nabla_a, \nabla_b] = H_{ab}{}^c d_c - \left(B_{ab}{}^c - B_{ba}{}^c \right) d_c + d_a B_b - d_b B_a + [B_a, B_b]. \tag{15}$$

If we introduce the tensor

$$T_{ab}{}^c = B_{ab}{}^c - B_{ba}{}^c - H_{ab}{}^c, \tag{16}$$

then we can rewrite F_{ab} as

$$F_{ab} = -T_{ab}{}^c \nabla_c + \frac{i}{4} R^{cd}{}_{ab} \Sigma_{cd}, \tag{17}$$

where $R^{cd}{}_{ab}$ has the expression

$$R^{cd}{}_{ab} = d_a B_b{}^{cd} - d_b B_a{}^{cd} + B_a{}^{de} B_{be}{}^c - B_b{}^{de} B_{ae}{}^c - H_{ab}{}^e B_e{}^{dc}. \tag{18}$$

In what follows we will use the shorthand notation

$$R_{ab} = \frac{i}{4} R^{cd}{}_{ab} \Sigma_{cd}. \tag{19}$$

As F_{ab} in (17) has a decomposition with respect to ∇_a and Σ_{cd} it acts in general not only as a matrix but also as a first order differential operator in field space. But, if we suppose that

$$H_{ab}{}^{c} = B_{ab}{}^{c} - B_{ba}{}^{c}, \tag{20}$$

that is we take $T_{ab}{}^{c} = 0$, then we can write Eq. (15) under the form:

$$F_{ab} = \frac{i}{4} R^{cd}{}_{ab} \Sigma_{cd} = R_{ab}. \tag{21}$$

We can verify that $T_{ab}{}^{c}$ and $R^{cd}{}_{ab}$ transform homogeneous under infinitesimal local DS gauge transformations. Then, as a consequence, the choice $T_{ab}{}^{c} = 0$ is indeed a gauge covariant statement as a implicitly assumed above.

3 REGULARIZATION

In order to analyze the regularization of our DS gauge theory, we will consider first the globally DS invariant action for a Dirac spinor field (matter field):

$$S_D = \int d^4x \left[\frac{i}{2} \overline{\psi} \gamma^a \left(\partial_a \psi \right) - \frac{i}{2} \left(\overline{\partial_a \psi} \right) \gamma^a \psi - m \overline{\psi} \psi \right]. \tag{22}$$

Then, if we want to obtain a gauge (local) invariant action, we have to change the usual derivative ∂_a in (22) by the gauge covariant derivative defined in Eq. (12):

$$S_D = \int d^4x \, e^{-1} \left[\frac{i}{2} \overline{\psi} \gamma^a \left(\nabla_a \psi \right) - \frac{i}{2} \left(\overline{\nabla_a \psi} \right) \gamma^a \psi - m \overline{\psi} \psi \right] \tag{23}$$

and to use the new volume element $d^4x \, e^{-1}$, where $e^{-1} = \det \left(\overline{e}_a{}^{b} \right)$.Then, partially integrating ∇_a in the second term of (23), we obtain the usual form of the Dirac action:

$$S_D = \int d^4x \, e^{-1} \overline{\psi} \left(i \gamma^a \nabla_a - m \right) \psi, \tag{24}$$

where we used the choice $T_{ab}{}^{c} = 0$.

The assumption that the interaction of the DS gauge fields with the matter fields (in our case with the Dirac field) can be regularized, imposes strong conditions on the classical gauge field dynamics. Namely, we know that the change of the partition function of the whole system under rescaling can be absorbed in its classical action yielding at most a nontrivial scale dependence of the different couplings, masses and wave function regularizations. As a consequence, the change of one-loop matter partition under rescaling will allow us to constrain the classical gauge field dynamics. The contribution of the

Dirac field to the partition function is given by the following functional integral [4]:

$$Z_{\psi}\left(e,B\right)=\int D\overline{\psi}D\psi e^{iS_{D}\left(\overline{\psi},\psi;e,B\right)}. \tag{25}$$

Then, we may perform a formal Grassmann integral in (25) and obtain:

$$Z_{\psi}\left(e,B\right)=e^{\frac{1}{2}\ln\det M_{\psi}\left(e,B\right)}, \tag{26}$$

where

$$M_{\psi}\left(e,B\right)=-D_{a}D^{a}+\frac{i}{2}R_{ab}\Sigma^{ab}-m^{2}. \tag{27}$$

Here, $M_{\psi}\left(e,B\right)$ is named hyperbolic fluctuation operator and its expression in (27) is obtain as usually [4] by squaring the Dirac operator introduced in Eq. (24). For the case ${T_{ab}}^{c}=0$ we consider here, the operator D_{a} in Eq. (27) is given by the formula:

$$D_{a}=\nabla_{a}+B_{a}. \tag{28}$$

The gauge field (Lie algebra valued) shall only act on the spinor indices and the covariant derivative ∇_{a} only on vector indices.

The spinor contribution to the partition function regularized at scale μ is given then by [5]:

$$Z_{\psi}\left(\mu;e,B\right)=e^{-\frac{1}{2}\zeta'\left(0;\mu;M_{\psi}\left(e,B\right)\right)}, \tag{29}$$

where $\zeta\left(s;\mu;M_{\psi}\left(e,B\right)\right)$ is the generalized zeta function of parameter s associated to the hyperbolic fluctuation operator $M_{\psi}\left(e,B\right)$ and $\zeta'\left(0;\mu;M_{\psi}\left(e,B\right)\right)$ is the derivative of the generalized zeta function with respect to s taken for $s=0$.

We consider now a new scale $\tilde{\mu}=\lambda\mu$ and determine the corresponding change of $Z_{\psi}\left(\mu;e,B\right)$. To end this, we use the very well known property [5]

$$\zeta'\left(0;\tilde{\mu};M_{\psi}\left(e,B\right)\right)=\zeta'\left(0;\mu;M_{\psi}\left(e,B\right)\right)+2\ln\lambda\cdot\zeta\left(0;\mu;M_{\psi}\left(e,B\right)\right). \tag{30}$$

Then, we obtain:

$$Z_{\psi}\left(\tilde{\mu};e,B\right)=Z_{\psi}\left(\mu;e,B\right)e^{-\ln\lambda\cdot\zeta\left(0;\mu;M_{\psi}\left(e,B\right)\right)}. \tag{31}$$

In order to obtain zeta function $\zeta\left(0,\mu,M_{\psi}\left(e,B\right)\right)$ we start with heat kernel equation

$$\left(\frac{\partial}{\partial\left(is\right)}+M_{x}\right)K\left(is;x,y\right), \tag{32}$$

together with an asymptotically s-expansion for the heat kernel $K(is;x,y)$ of the form

$$K(is;x,y) \propto \frac{i}{4\pi i s^{d/2}} \exp\left(-\frac{r^2(x,y)}{4is}\right) \sum_{k=0}^{\infty} (is)^k c_k(x,y). \tag{33}$$

In eq. (32) the differential operator M acts on heat kernel K and the index x denotes the derivative of heat kernel with respect to x.

We remember the fact that zeta function in four dimensions is given by

$$\zeta(0;\mu;M_\psi(e,B)) = \frac{i}{(4\pi)^2} \int d^4x \det e^{-1} tr c_2(x) \tag{34}$$

and the coefficient function $c_2(x)$ for the Dirac field in the case $T=0$ has the form

$$\begin{aligned} tr_D c_2 &= \frac{1}{30}\nabla_c{}^c R^{ab}{}_{ab} + \frac{1}{72} R_{ab}{}^{ab} \cdot R_{cd}{}^{cd} \\ &- \frac{7}{360} R_{abcd} \cdot R^{abcd} - \frac{1}{45} R_{ac}{}^a{}_d \cdot R_b{}^{cbd} + \\ &+ \frac{1}{3} m^2 \cdot R^{ab}{}_{ab} + 2m^4. \end{aligned} \tag{35}$$

Using equations (34) and (35) we obtain the value of $\zeta(0,\mu,M_\psi(e,B))$ as an integral over $tr_D c_2$. Next, introducing the value of $\zeta(0,\mu,M_\psi(e,B))$ in the partition function (31) allow us to obtain the anomalous terms in the spinorial case and then to determine a minimal field gauge action compatible with regularization requirements. Regularization of any theory, including dynamical gauge fields, requires that these contributions to the partition function like (31) be expressed as local DS gauge invariant polynomials in the fields $e_a{}^b$ and $B_a{}^{bc}$. In our case, under the constraint $T_{ab}{}^c = 0$, we obtain as minimal classical action for the gauge fields:

$$\begin{aligned} & S_{gauge}(e,B) = \\ = & \int d^4x\, e^{-1} \left(-\frac{1}{16\pi G}(R-2\Lambda) + \alpha R^2 + \beta R_{ac}{}^a{}_d R_b{}^{cbd} + \gamma R_{abcd} R^{abcd}\right). \end{aligned} \tag{36}$$

Here, G is the gravitational constant and α, β, γ are the coupling constants. We can see that the DS gauge group automatically enforces a cosmological constant which in our model is equal to $\Lambda = -12\lambda^2$, where λ is the deformation parameter of the de Sitter Lie algebra. We emphasize that S_{gauge} in (36) is an action for gauge fields defined on the Minkowski space-time (M_4, η) and is invariant on one hand under local DS gauge transformations, on the other hand under global Poincaré symmetry reflecting the symmetry of the underlying space-time.

4 CONCLUSION

Based on the hypothesis that DS is a purely inner symmetry we have developed a gauge theory of gravitation with the constant cosmological automatically included. When the deformation parameter $\lambda \to 0$, we obtain the Poincaré gauge theory on the Minkowski space-time which do not include the cosmological constant. The gravitational interaction is mediated by gauge fields defined on a fixed Minkowski space-time. Their dynamics has been determined imposing consistency requirements with regularization properties of matter fields in the gravitational backgrounds. In our model there is no any direct interrelation between gravity and the structure of spacetime. At quantum level it may conceptually be easier to deal with a field theoretical description of gravitation free of any geometrical aspects.

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