NONCOMMUTATIVE CORRECTIONS TO SPHERICALLY SYMMETRIC GRAVITATIONAL FIELDS

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Abstract. We develop a noncommutative theory of gravity by gauging the noncommutative SO(4,1) de-Sitter group and using the Seiberg-Witten map, with subsequent contraction to the Poincaré (inhomogeneous Lorentz) group ISO(3,1). The gauge fields are assumed to have spherical symmetry and the commutative torsion is constrained to vanish. The gauge fields (tetrads) are determined up to the second order in the noncommutative parameter. A deformed real metric is then defined and its components are obtained. As an application we calculate the noncommutative corrections to the Schwarzschild solution. The possibility to obtain such corrections for Reissner-Nordström metric is also discussed. Some of the implications of the noncommutative Schwarzschild and Reissner-Nordström metrics in connection to their singularity structure and Black-Hole physics are also mentioned.

1. Introduction

It is well known that the noncommutativity of space-time is one of the presently option for describing the quantum properties of matter at very high energy scale.. If the nature has chosen such a course, it is most sensible to search for manifestations of the noncommutativity of space-time at the "natural laboratories" of the highest energy, i.e. the gravitational singularities.

The noncommutativity of space-time is intrinsically connected with gravity [1, 2]. Gauge theories of gravitation have been intensively studied up to now, both on commutative [3-4] (see also the reviews in [5,6]) [7] and noncommutative [8, 9] space-times. Many of recent researches are orientated toward a formulation of General Relativity on noncommutative space-times. In Ref. [8] for example, a deformation of Einstein's gravity was studied by gauging the noncommutative SO(4,1) de-Sitter group and using the Seiberg-Witten map [2, 10, 11] with subsequent contraction to the Poincaré (inhomogeneous Lorentz) group ISO(3,1). Another construction of noncommutative gravitational theory was proposed in Refs. [12], and it is based on the twisted Poincaré algebra [13]. The twisting procedure insures the Lorentz invariance of the algebra $\left[x^{\mu}, x^{\nu}\right] = i\theta^{\mu\nu}$ (canonical structure) defining the noncommutativity of the space-time. However, it has been shown that the dynamics of the noncommutative gravity coming from string theory [14] is much richer that one in this version of deformed gravity [1]. Ref. [15] contains results on noncommutative General Relativity for a restrictive class of coordinate transformations which preserve the canonical structure. By gauging the Lorentz algebra *so*(*3,1*) within the enveloping algebra approach one obtains a theory of noncommutative General Relativity restricted to the volume-preserving transformations (unimodular theory of gravity). Another attempted approach was to twist the gauge Poincaré algebra [16]. It is worthwhile to emphasize that there remains one more important unsolved problem in all these theories: to establish a Leibniz rule for gauge transformations of fields [17, 18], since the star product is not invariant under the diffeomorphism transformations. Steps towards this goal have been taken in a geometrical approach to noncommutative gravity [19].

The investigation of noncommutative gauge theories of gravitation is also motivated by the possibility of their applications to the physics of quantum black holes [20,21]. It could provide a satisfactory description of black holes in those extreme regimes, where stringy effects are considered relevant [22, 23].

It is known that the black hole characteristic quantities depend on the Hawking temperature via the usual thermodynamically relations. The Hawking temperature undergoes corrections from many sources: the quantum corrections [24], the self-gravitational corrections [25], and the corrections due to the generalized uncertainty principle [26]. But, it is possible to have also relevant corrections due to the space-time noncommutativity. We will give such corrections for the cases of Schwarzschild and Reissner-Nordström black holes.

In this paper we present a deformed Schwarzschild solution in noncommutative gauge theory of gravitation proceeding along the approach in Ref. [8]. Although this version of noncommutative version is certainly not a final one, we believe that the complete theory will retain the main features of this approach. Partly, the results referring to Schwarzschild solution are contained in our recent paper [28]. First, we develop a de-Sitter gauge theory of gravitation over a spherical symmetric commutative Minkowski space-time [7]. Then, a deformation of the gravitational field is constructed by gauging the noncommutative de-Sitter SO(4,1) group [8] and using Seiberg-Witten map [2]. The space-time of noncommutative theory will be also of Minkowski type but it will be endowed with spherical noncommutative coordinates. The deformed gauge fields are determined up to the second order in the noncommutativity parameters $\theta^{\mu\nu}$.

Finally, the deformed gravitational gauge potentials (tetrad fields) $\hat{e}_{\mu}^{a}(x,\theta)$ are obtained by contraction of the noncommutative gauge group SO(4,1) to the Poincaré (inhomogeneous Lorentz) group ISO(3,1). As an application, we calculate these potentials for the case of a Schwarzschild solution and define the corresponding deformed metric $\hat{g}_{\mu\nu}(x,\theta)$. It is for the first time when such a deformed metric is given for a 4-dimensional noncommutative space-time. The corrections appear only in the second order of the expansion in θ, i.e. there are no any first order ones. We will give also an evaluation of the noncommutativity corrections to the red shift test of General Relativity. The conclusion is that the value of this correction is with about 18 orders less than that resulting from General Relativity in the case of Sun. Therefore, it is presently impossible to verify experimentally the noncommutativity correction to the red shift test of General Relativity.

The calculations are very laborious, so that we used an analytical program conceived in GRTensor II package for the Maple platform. Specific routines have been written and adapted for Maple system.

Section 2 is devoted to the commutative gauge theory of the de-Sitter group SO(4,1) formulated on a 4-dimensional Minkowski space-time endowed with s spherical metric. The Section 3 contains the results regarding the noncommutative theory. The deformed gauge potentials (tetrad fields) are obtained up to the second order of the expansion in θ. Based on these results, we define a deformed real metric and calculate its components in the case of a Schwarzschild solution. Using the results we determine in Section 4 the deformed Schwarzschild metric. The corrections are obtained up to the second order of the noncommutativity parameters $\theta^{\mu\nu}$. An evaluation of the value for the correction to the red shift test of General Relativity is also given. In section 5 we present preliminary results on the noncommutativity corrections for the case of the Reissner-Nordström metric. The possibility of obtain corrections for the thermodynamical quantities like the horizon radius, the Hawking temperature and the entropy of a black hole is also investigated. Some concluding remarks are given in Section 6.

2. Commutative gauge theory

We review first the gauge theory of the de-Sitter group SO(4,1) on a commutative 4-dimensional Minkowski space-time endowed with the spherical symmetric metric [7]:

$$ds^2 = dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right) - c^2\, dt^2 . \qquad (2.1)$$

This means that coordinates on this space-time are chosen as $\left(x^\mu\right) = (r,\theta,\varphi,ct), \mu = 1,2,3,0$. The SO(4,1) group is 10-dimensional and its infinitesimal generators are denoted by $M_{AB} = -M_{BA}, A,B = 1,2,3,0,5$. If we introduce indices $a,b,\dots = 1,2,3,0$, i.e. we put $A = a,5$, $B = b,5$, etc., then the generators M_{AB} can be identified with translations $P_a = M_{a5}$ and Lorentz rotations $M_{ab} = -M_{ba}$. The corresponding non-deformed gauge potentials will be denoted by $\omega_\mu^{AB}(x) = -\omega_\mu^{BA}(x)$. They are identified with $\omega_\mu^{ab}(x) = -\omega_\mu^{ba}(x)$ (spin connection) and $\omega_\mu^{a5}(x) = k\, e_\mu^a(x)$ (tetrad fields), where k is the contraction parameter. For the limit $k \to 0$ we obtain the ISO(3,1) gauge group, i.e., the commutative Poincaré gauge theory of gravitation. The strength field associated to $\omega_\mu^{AB}(x)$ is [7]:

$$F_\mu^{AB} = \partial_\mu \omega_\nu^{AB} - \partial_\nu \omega_\mu^{AB} + \left(\omega_\mu^{AC}\,\omega_\nu^{DB} - \omega_\nu^{AC}\,\omega_\mu^{DB}\right)\eta_{CD}\,, \qquad (2.2)$$

where $\eta_{AB} = diag(1,1,1,-1,1)$. Then, we have:

$$F_{\mu\nu}^{a5} \equiv k\,T_{\mu\nu}^a = k\left[\partial_\mu e_\nu^a - \partial_\nu e_\mu^a + \left(\omega_\mu^{ab}\, e_\nu^c - \omega_\nu^{ab}\, e_\mu^c\right)\eta_{bc}\right], \qquad (2.3)$$

$$F_{\mu\nu}^{ab} \equiv R_{\mu\nu}^{ab} = \partial_\mu \omega_\nu^{ab} - \partial_\nu \omega_\mu^{ab} + \left(\omega_\mu^{ac}\omega_\nu^{db} - \omega_\nu^{ac}\omega_\mu^{db}\right)\eta_{cd} + k\left(e_\mu^a e_\nu^b - e_\nu^a e_\mu^b\right), \qquad (2.4)$$

where $\eta_{ab} = diag(1,1,1,-1)$. The Poincare gauge theory that we are using has the geometric structure of the Riemann-Cartan space-time U(4) with curvature and torsion [6]. The quantity $T_{\mu\nu}^a$ is interpreted as the torsion tensor and $R_{\mu\nu}^{ab}$ as the curvature tensor of a

Riemann-Cartan space-time defined by the gravitational fields $e_\mu^a(x)$ and $\omega_\mu^{ab}(x)$. By imposing the condition of null torsion $T_{\mu\nu}^a = 0$, one can solve for $\omega_\mu^{ab}(x)$ in terms of $e_\mu^a(x)$, i.e. the spin connection components are determined by tetrads (they are not independent fields).

Now, we consider a particular form of spherically gauge fields of the SO(4,1) group given by the following ansatz [7]:

$$e_\mu^1 = \left(\frac{1}{A},0,0,0\right), e_\mu^2 = (0,r,0,0),\ e_\mu^3 = (0,0,r\sin\theta,0)\ ,\ e_\mu^0 = (0,0,0,A), \quad (2.5)$$

$$\begin{aligned}&\omega_\mu^{12} = (0,W,0,0), \omega_\mu^{13} = (0,0,\mathrm{Z}\sin\theta,0), \omega_\mu^{23} = (0,0,-\cos\theta,\mathrm{V}),\\ &\omega_\mu^{10} = (0,0,0,U), \omega_\mu^{20} = \omega_\mu^{30} = (0,0,0,0),\end{aligned} \quad (2.6)$$

where A, U, V, W and Z are functions only of the three-dimensional radius (r). The non-null components of $T_{\mu\nu}^a$ and $R_{\mu\nu}^{ab}$ are [4]:

$$\begin{aligned}&T_{01}^0 = -\frac{AA'+U}{A}, T_{03}^2 = rV\sin\theta, T_{12}^2 = \frac{A+W}{A},\\ &T_{02}^3 = -rV, T_{13}^3 = \frac{(A+Z)\sin\theta}{\mathrm{A}},\end{aligned} \quad (2.7)$$

and respectively

$$\begin{aligned}&R_{01}^{01} = U', R_{01}^{23} = -V', R_{23}^{13} = (Z-W)\cos\theta,\\ &R_{02}^{02} = -UW, R_{02}^{13} = -VW, R_{03}^{03} = -UZ\sin\theta,\\ &R_{03}^{12} = VZ\sin\theta, R_{12}^{12} = W', R_{23}^{23} = (1-ZW)\sin\theta,\\ &R_{13}^{13} = Z'\sin\theta,\end{aligned} \quad (2.8)$$

where A', U', V', W' and Z' denote the derivatives with respect to the r-coordinate.

If we use (2.7), then the condition of null-torsion gives the following constraints:

$$U = -AA', V = 0, W = Z = -A, \quad (2.9)$$

as we already have mentioned. Then, from the field equations for $e_\mu^a(x)$

$$R_\mu^a - \frac{1}{2}Re_\mu^a = 0,\ R_\mu^a = R_{\mu\nu}^{ab}\,\bar{e}_b^\nu, R = R_{\mu\nu}^{ab}\,\bar{e}_a^\mu\,\bar{e}_b^\nu, \quad (2.10)$$

$\bar{e}_a^\mu$ being the inverse of e_μ^a, we obtain the solution [7]

$$A^2 = 1 - \frac{\alpha}{\mathrm{r}}, \quad (2.11)$$

where α is an arbitrary constant of integration. For $\alpha = \frac{2GM}{c^2}$ we obtain the commutative Schwarzschild solution (G is the Newton constant and M is the mass of the point-like source of the gravitational field). The corresponding metric

$$g_{\mu\nu} = \eta_{\mathrm{ab}}\,e_\mu^a\,e_\nu^b, \quad (2.12)$$

has the following non-null components

$$g_{11}=\frac{1}{1-\frac{2GM}{c^2 r}},\, g_{22}=\frac{g_{33}}{\sin\theta}=r,\, g_{00}=-\left(1-\frac{2GM}{c^2 r}\right). \tag{2.13}$$

We emphasize that this solution is obtained from the commutative SO(4,1) gauge theory with a contraction $k \to 0$ to the Poincaré group ISO(3,1).

We will follow now the Ref, [8] in order to obtain a deformation of gravitation by gauging the noncommutative de-Sitter SO(4,1) group. Then, by contraction to the Poincaré (inhomogeneous Lorentz) group ISO(3,1) we will obtain the deformed gauge fields $\hat{e}^a_\mu(x,\theta)$. In the next three Sections we will calculate these fields for the case of a Schwarzschild and Reissner-Nordström solutions and define the corresponding deformed metrics $\hat{g}_{\mu\nu}(x,\theta)$ up to the second order of the expansion in θ.

3. Deformed gauge fields

We suppose that the noncommutative structure of the space-time is determined by

$$\left[x^\mu, x^\nu\right]=i\theta^{\mu\nu}, \tag{3.1}$$

where $\theta^{\mu\nu}=-\theta^{\nu\mu}$ are constant (canonical) parameters. To develop the noncommutative gauge theory we introduce the star "*" product between the functions *f* and *g* defined over this space-time:

$$(f*g)(x)=f(x)e^{\frac{i}{2}\theta^{\mu\nu}\overleftarrow{\partial}_\mu\otimes\overrightarrow{\partial}_\nu}g(x). \tag{3.2}$$

The gauge fields for the noncommutative case are denoted by $\hat{\omega}^{AB}_\mu(x,\theta)$,and they are subjected to the reality constraints [8, 10, 11]:

$$\begin{aligned}&\hat{\omega}^{AB+}_\mu(x,\theta)=-\hat{\omega}^{BA}_\mu(x,\theta),\\ &\hat{\omega}^{AB}_\mu(x,\theta)^r\equiv\hat{\omega}^{AB}_\mu(x,-\theta)=-\hat{\omega}^{BA}_\mu(x,\theta).\end{aligned} \tag{3.3}$$

with "+" denoting the complex conjugate

By expanding $\hat{\omega}^{ab}_\mu(x,\theta)$in powers of noncommutative parameter θ:

$$\hat{\omega}^{AB}_\mu(x,\theta)=\omega^{AB}_\mu(x)-i\theta^{\nu\rho}\,\omega^{AB}_{\mu\nu\rho}(x)+\theta^{\nu\rho}\theta^{\lambda\tau}\,\omega^{AB}_{\mu\nu\rho\lambda\tau}(x)+\cdots, \tag{3.4}$$

then the constraints (3.1) imply the properties

$$\omega^{AB}_\mu(x)=-\omega^{BA}_\mu(x),\ \omega^{AB}_{\mu\nu\rho}(x)=\omega^{BA}_{\mu\nu\rho}(x),\ \omega^{AB}_{\mu\nu\rho\lambda\tau}(x)=-\omega^{BA}_{\mu\nu\rho\lambda\tau}(x),\ \cdots \tag{3.5}$$

Using the Seiberg-Witten map [2] one obtains the following noncommutative corrections up to the second order [8]:

$$\omega^{\mathrm{AB}}_{\mu\nu\rho}(x)=\frac{1}{4}\left\{\omega_\nu,\partial_\rho\omega_\mu+R_{\rho\mu}\right\}^{AB}, \tag{3.6}$$

$$\begin{aligned}\omega^{\mathrm{AB}}_{\mu\nu\rho\lambda\tau}(x)=\frac{1}{32}\Big(&-\left\{\omega_\lambda,\partial_\tau\left\{\omega_\nu,\partial_\rho\omega_\mu+R_{\rho\mu}\right\}\right\}+2\left\{\omega_\lambda,\left\{R_{\tau\nu},R_{\mu\rho}\right\}\right\}-\\ &-\left\{\omega_\lambda,\left\{\omega_\nu,D_\rho R_{\tau\mu}+\partial_\rho R_{\tau\mu}\right\}\right\}-\left\{\left\{\omega_\nu,\partial_\rho\omega_\lambda+R_{\rho\lambda}\right\},\left(\partial_\tau\omega_\mu+R_{\tau\mu}\right)\right\}+\\ &+2\left[\partial_\nu\omega_\lambda,\partial_\rho\left(\partial_\tau\omega_\mu+R_{\tau\mu}\right)\right]\Big)^{AB}\end{aligned} \tag{3.7}$$

where

$$\{\alpha,\beta\}^{AB}=\alpha^{AC}\,\beta_C{}^B+\beta^{AC}\,\alpha_C{}^B,\ [\alpha,\beta]^{AB}=\alpha^{AC}\,\beta_C{}^B-\beta^{AC}\,\alpha_C{}^B. \tag{3.8}$$

and

$$D_\mu R^{AB}_{\rho\sigma}=\partial_\mu R^{AB}_{\rho\sigma}+\left(\omega^{AC}_\mu R^{DB}_{\rho\sigma}+\omega^{BC}_\mu R^{DA}_{\rho\sigma}\right)\eta_{CD}. \tag{3.9}$$

As in the commutative case, we write $\hat{\omega}^{a5}_\mu(x,\theta)=k\,\hat{e}^a_\mu(x,\theta)$ and $\hat{\omega}^{55}_\mu(x,\theta)=k\,\phi_\mu(x,\theta)$. Then, we impose the condition of null torsion $T^a_{\mu\nu}=0$ and not $\hat{T}^a_{\mu\nu}=0$ because by contraction $k\to 0$ the quantity $\phi_\mu(x,\theta)$ will drop out [8]. Using (3.6) and (3.7) we obtain, in the limit $k\to 0$, the deformed tetrad fields $\hat{e}^a_\mu(x,\theta)$ up to the second order:

$$\hat{e}^a_\mu(x,\theta)=e^a_\mu(x)+i\,\theta^{\nu\rho}\,e^a_{\mu\nu\rho}(x)+\theta^{\nu\rho}\,\theta^{\lambda\tau}\,e^a_{\mu\nu\rho\lambda\tau}(x)+O\left(\theta^3\right), \tag{3.10}$$

where

$$e^a_{\mu\nu\rho}=-\frac{1}{4}\left[\omega^{ac}_\nu\partial_\rho e^d_\mu+\left(\partial_\rho\omega^{ac}_\mu+R^{ac}_{\rho\mu}\right)e^d_\nu\right]\eta_{cd}, \tag{3.11}$$

$$\begin{aligned}e^a_{\mu\nu\rho\lambda\tau}=\frac{1}{32}\Big[&2\{R_{\tau\nu},R_{\mu\rho}\}^{ab}e^c_\lambda-\omega^{ab}_\lambda\left(D_\rho R^{cd}_{\tau\mu}+\partial_\rho R^{cd}_{\tau\mu}\right)e^m_\nu\,\eta_{dm}-\\&-\{\omega_\nu,\left(D_\rho R_{\tau\mu}+\partial_\rho R_{\tau\mu}\right)\}^{ab}e^c_\lambda-\partial_\tau\{\omega_\nu,\left(\partial_\rho\omega_\mu+R_{\rho\mu}\right)\}^{ab}e^c_\lambda-\\&-\omega^{ab}_\lambda\partial_\tau\left(\omega^{cd}_\nu\partial_\rho e^m_\mu+\left(\partial_\rho\omega^{cd}_\mu+R^{cd}_{\rho\mu}\right)e^m_\nu\right)\eta_{dm}+2\partial_\nu\omega^{ab}_\lambda\partial_\rho\partial_\tau e^c_\mu-\\&-2\partial_\rho\left(\partial_\tau\omega^{ab}_\mu+R^{ab}_{\tau\mu}\right)\partial_\nu e^c_\lambda-\{\omega_\nu,\left(\partial_\rho\omega_\lambda+R_{\rho\lambda}\right)\}^{ab}\partial_\tau e^c_\mu-\\&-\left(\partial_\tau\omega^{ab}_\mu+R^{ab}_{\tau\mu}\right)\left(\omega^{cd}_\nu\partial_\rho e^m_\lambda+\left(\partial_\rho\omega^{cd}_\lambda+R^{cd}_{\rho\lambda}\right)e^m_\nu\,\eta_{dm}\right)\Big]\eta_{bc}.\end{aligned} \tag{3.12}$$

We define also the complex conjugate of the deformed tetrad fields $\hat{e}^a_\mu(x,\theta)$ given in (3.10) by:

$$\hat{e}^{a+}_\mu(x,\theta)=e^a_\mu(x)-i\,\theta^{\nu\rho}\,e^a_{\mu\nu\rho}(x)+\theta^{\nu\rho}\,\theta^{\lambda\tau}\,e^a_{\mu\nu\rho\lambda\tau}(x)+O\left(\theta^3\right). \tag{3.13}$$

Then, we can introduce a deformed metric by formula:

$$\hat{g}_{\mu\nu}(x,\theta)=\frac{1}{2}\eta_{ab}\left(\hat{e}^a_\mu*\hat{e}^{b+}_\nu+\hat{e}^b_\mu*\hat{e}^{a+}_\nu\right). \tag{3.14}$$

We can see that this metric is, by definition, a real one, even if the deformed tetrad fields $\hat{e}^a_\mu(x,\theta)$ are complex quantities.

4. Second order corrections to Schwarzschild solution

Using the ansatz (2.5) – (2.6), we can determine the deformed Schwarzschild metric. To end this, we have to obtain first the corresponding components of the tetrad fields $\hat{e}^a_\mu(x,\theta)$ and their complex conjugated $\hat{e}^{a+}_\mu(x,\theta)$ given by the Eqs. (3.10) and (3.13). With the definition (3.14) it is possible then to obtain the components of the deformed metric $\hat{g}_{\mu\nu}(x,\theta)$.

To simplify the calculations, we choose the parameters $\theta^{\mu\nu}$ as:

$$\theta^{\mu\nu} = \begin{pmatrix} 0 & \theta & 0 & 0 \\ -\theta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mu,\nu = 1,2,3,0. \tag{4.1}$$

Here the constant quantity θ, which determine noncommutativity of the space-time coordinates, is chosen so that it has the dimension L^2 (square of length).

The non-null components of tetrad fields $\hat{e}^{a}_{\mu}(x,\theta)$ are:

$$\hat{e}^1_1 = \frac{1}{A} + \frac{A''}{8}\theta^2 + O(\theta^3), \tag{4.2a}$$

$$\hat{e}^1_2 = -\frac{i}{4}(A + 2r A')\,\theta + O(\theta^3), \tag{4.2b}$$

$$\hat{e}^2_2 = r + \frac{1}{32}\left(7 A A' + 12 r A'^2 + 12 r A A''\right)\theta^2 + O(\theta^3), \tag{4.2c}$$

$$\hat{e}^3_3 = r\sin\theta - \frac{i}{4}(\cos\theta)\theta + \frac{1}{8}\left(2 r A'^2 + r A A'' + 2 A A' - \frac{A'}{A}\right)(\sin\theta)\theta^2 + O(\theta^3), \tag{4.2d}$$

$$\hat{e}^0_0 = A + \frac{1}{8}\left(2 r A'^3 + 5 r A A' A'' + r A^2 A''' + 2 A A'^2 + A^2 A''\right)\theta^2 + O(\theta^3). \tag{4.2e}$$

where A', A'', A''' are respectively first, second and third derivatives of $A(r)$. The complex conjugated components can be easily obtained from these expressions.

Then, using the definition (3.14), we obtain the following non-null components of the deformed metric up to the second order:

$$\hat{g}_{11}(x,\theta) = \frac{1}{A^2} + \frac{1}{4}\frac{A''}{A}\theta^2 + O(\theta^4), \tag{4.3}$$

$$\hat{g}_{22}(x,\theta) = r^2 + \frac{1}{16}\left(A^2 + 11 r A A' + 16 r^2 A'^2 + 12 r^2 A A''\right)\theta^2 + O(\theta^4),$$

$$\hat{g}_{33}(x,\theta) = r^2\sin^2\theta + \frac{1}{16}\left[4\left(2 r A A' - r\frac{A'}{A} + r^2 A A'' + 2 r^2 A'^2\right)\sin^2\theta + \cos^2\theta\right]\theta^2 + O(\theta^4)$$

$$\hat{g}_{00}(x,\theta) = -A^2 - \frac{1}{4}\left(2 r A A'^3 + r A^3 A''' + A^3 A'' + 2 A^2 A'^2 + 5 r A^2 A' A''\right)\theta^2 + O(\theta^4),$$

For $\theta \to 0$ we obtain the commutative Schwarzschild solution with $A^2 = 1 - \frac{\alpha}{r}$ [see Eq. (2.11)].

It is interesting to remark that, if we choose the parameters $\theta^{\mu\nu}$ as in (4.1), then the deformed metric $\hat{g}_{\mu\nu}(x,\theta)$ is diagonal as it is in the commutative case. But, in general, for arbitrary $\theta^{\mu\nu}$, the deformed metric $\hat{g}_{\mu\nu}(x,\theta)$ is not diagonal even if the commutative (non-deformed) one has this property. Therefore, we can conclude that the noncommutativity modifies the structure of the gravitational field.

For the Schwarzschild solution we have:

$$A(r)=\sqrt{1-\frac{\alpha}{r}},\ \ \alpha=\frac{2GM}{c^2}; \tag{4.4}$$

The function $A(r)$ is non-dimensional, but its derivatives A', A'' and A''' have respectively the dimensions L^{-1} L^{-2} and L^{-3}. As a consequence, all the components of the deformed metric $\hat{g}_{\mu\nu}(x,\theta)$ in (4.3) have the correct dimensions.

Now, if we introduce (4.4) in (4.3), then we obtain the deformed Schwarzschild metric. Its non-null components are:

$$\begin{aligned}
\hat{g}_{11} &= \frac{1}{1-\frac{\alpha}{r}}-\frac{\alpha(4\mathrm{r}-3\alpha)}{16r^2(\mathrm{r}-\alpha)^2}\theta^2+O(\theta^4),\\
\hat{g}_{22} &= r^2+\frac{2\mathrm{r}^2-17\alpha\mathrm{r}+17\alpha^2}{32r(\mathrm{r}-\alpha)}\theta^2+O(\theta^4),\\
\hat{g}_{33} &= r^2\sin^2\theta+\frac{\left(\mathrm{r}^2+\alpha\mathrm{r}-\alpha^2\right)\cos^2\theta-\alpha(2\mathrm{r}-\alpha)}{16r(\mathrm{r}-\alpha)}\theta^2+O(\theta^4),\\
\hat{g}_{00} &= -\left(1-\frac{\alpha}{r}\right)-\frac{\alpha(8\mathrm{r}-11\alpha)}{16r^4}\theta^2+O(\theta^4).
\end{aligned} \tag{4.5}$$

We can evaluate then the contributions of these corrections to the tests of General Relativity. For example, if we consider the red shift of the light propagating in a gravitational field [27], then we obtain for the case of the Sun:

$$\frac{\Delta\lambda}{\lambda}=\frac{\alpha}{2\mathrm{R}}-\frac{\alpha(8\mathrm{R}-11\alpha)}{32\mathrm{R}^4}\theta^2+O\left(\theta^4\right), \tag{4.6}$$

where R is the radius of the Sun. Because for the Sun we have $\alpha=\frac{2GM}{c^2}=2.95\cdot 10^3\ m$ and $R=6.955\cdot 10^8\ m$, then we obtain from (4.5):

$$\frac{\Delta\lambda}{\lambda}=2\cdot 10^{-6}-2.19\cdot 10^{-24}\,\theta^2+O\left(\theta^4\right). \tag{4.7}$$

The noncommutativity correction has a value that is with about 18 orders less than that which result from General Relativity. Therefore, presently it is not possible to verify experimentally the noncommutativity correction to the red shift test of General Relativity.

5. Corrections to the Reissner-Nordström solution

The results from previous Sections apply to any spherically gravitational fields having the gauge fields defined as in Eqs. (2.5) and (2.6). In particular, they can be used also for the Reissner-Nordström metric, with the function $A(r)$ given by:

$$A(r)=1-\frac{2m}{r}+\frac{Q^2}{r^2}, \tag{5.1}$$

where m is the mass and Q the electric charge of the point-like source of gravitational field. It is very easy to verify that from Eqs. (4.3) we obtain for $\theta\to 0$, the commutative Reissner-Nordström solution if we consider the expression (5.1).

Now, if we insert $A(r)$ from (5.1) into Eqs. (4.3), then we obtain the deformed Reissner-Nordström metric. Its non-null components are:

$$\hat{g}_{11}(r,\theta)=\frac{1}{1-\frac{2m}{r}+\frac{Q^2}{r^2}}+\frac{\left(-2mr^3+3m^2r^2+3Q^2r^2-6mQ^2r+2Q^6\right)\theta^2}{4r^2\left(r^2-2mr+Q^2\right)^2}, \qquad (5.2)$$

$$\hat{g}_{22}(r,\theta)=r^2+\frac{\left(r^4-17mr^3+34m^2r^2+27Q^2r^2-75mQ^2r+30Q^4\right)\theta^2}{16r^2\left(r^2-2mr+Q^2\right)}, \qquad (5.3)$$

$$\hat{g}_{33}(r,\theta)=r^2\sin^2\theta+\frac{\cos^2\theta\left(r^4+2mr^3-7Q^2r^2-4m^2r^2+16Q^2r-8Q^4\right)\theta^2}{16r^2\left(r^2-2mr+Q^2\right)}+$$
$$+\frac{\left(-4mr^3+8Q^2r^2+4m^2r^2-16mQ^2r+8Q^4\right)\theta^2}{16r^2\left(r^2-2mr+Q^2\right)}, \qquad (5.4)$$

$$\hat{g}_{00}(r,\theta)=-\left(1-\frac{2m}{r}+\frac{Q^2}{r^2}\right)-\frac{\left(4mr^3-9Q^2r^2-11m^2r^2+30mQ^2r-14Q^4\right)\theta^2}{4\mathrm{r}^6}. \qquad (5.5)$$

It will be very interesting to study the gravitational singularities of the deformed scalar curvature using these results. The work on this subject is in progress [29].

The expression (5.5) can be used to obtain the corrections to the thermodynamical quantities due to the space-time noncommutativity. If we consider the corrected event horizon radius up to the second order in θ as

$$\hat{r}=A+B\theta+C\theta^2, \qquad (5.6)$$

then we can obtain the unknown coefficients A, B, C by imposing the condition

$$\hat{g}_{00}(\hat{r},\theta)=0. \qquad (5.7)$$

Substituting (5.6) into (5.7) and using the expression (5.5), we obtain the following expression for the event horizon radius:

$$\hat{r}=m+\sqrt{m^2-Q^2}+$$
$$+\frac{\left(48m^4+\sqrt{m^2-Q^2}\left(48m^3-64mQ^2\right)-88Q^2m^2-Q^2+40Q^4\right)}{16\left(32m^5+\sqrt{m^2-Q^2}\left(32m^4-32m^2Q^2+4Q^4\right)-48m^3Q^2+16mQ^4\right)}\theta^2. \qquad (5.8)$$

The modified Hawking-Bekenstein temperature and the horizon area of Reissner-Nordström black hole in noncommutative space-time to the second order of θ are as following respectively:

$$\hat{T}=\frac{1}{4\pi}\frac{d\hat{g}_{00}(\hat{r},\theta)}{\mathrm{d\,r}}=\frac{m^2+m\sqrt{m^2-Q^2}}{2\pi\left(\mathrm{m}+\sqrt{m^2-Q^2}\right)^3}+$$
$$+\left(\frac{\left(448m^9-1648Q^2m^7+2112Q^4m^5-1091Q^6m^3+179Q^8m\right)}{16\pi\left(\mathrm{m}+\sqrt{m^2-Q^2}\right)^7\left[8m^5-12Q^2m^3+4Q^4m+\sqrt{m^2-Q^2}\left(8m^4-8Q^2m^2+Q^4\right)\right]}+\right.$$

$$+\frac{\sqrt{m^2-Q^2}\left(2240Q^2m^6+2597Q^4m^4+1053Q^6m^2+612m^8+84Q^8\right)}{16\pi\left(m+\sqrt{m^2-Q^2}\right)^7\left[8m^5-12Q^2m^3+4Q^4m+\sqrt{m^2-Q^2}\left(8m^4-8Q^2m^2+Q^4\right)\right]}+$$

$$+\frac{\left(m^2-Q^2\right)^{3/2}\left(264Q^2m^4-473Q^4m^2-152m^6+51Q^6\right)}{16\pi\left(m+\sqrt{m^2-Q^2}\right)^7\left[8m^5-12Q^2m^3+4Q^4m+\sqrt{m^2-Q^2}\left(8m^4-8Q^2m^2+Q^4\right)\right]}+$$

$$\left.+\frac{\left(m^2-Q^2\right)^{5/2}\left(16Q^2m^2-12m^4\right)}{16\pi\left(m+\sqrt{m^2-Q^2}\right)^7\left[8m^5-12Q^2m^3+4Q^4m+\sqrt{m^2-Q^2}\left(8m^4-8Q^2m^2+Q^4\right)\right]}\right)\theta^2 \quad (5.9)$$

$$\hat{A}=4\pi\hat{r}^2=4\pi\left(m+\sqrt{m^2-Q^2}\right)^2+$$
$$+\frac{\pi\left(m+\sqrt{m^2-Q^2}\left[6m^4-11Q^2m^2+5Q^4+\sqrt{m^2-Q^2}\left(6m^3-8Q^2m\right)\right]\right)}{8m^5-12Q^2m^3+4Q^4m+\sqrt{m^2-Q^2}\left(8m^4-8Q^2m^2+Q^4\right)}\theta^2 \quad (5.10)$$

According to the Hawking-Bekenstein formula , the thermodynamic entropy of a black hole is proportional to the area of the event horizon $S=\frac{A}{4}$, where A is the area of the horizon. The corrected entropy due to noncommutativity of space-time is:

$$\hat{S}=\frac{\hat{A}}{4}=\pi\left(m+\sqrt{m^2-Q^2}\right)^2+$$
$$+\frac{\pi\left(m+\sqrt{m^2-Q^2}\right)\left[6m^4-11Q^2m^2+5Q^4+\sqrt{m^2-Q^2}\left(6m^3-8Q^2m\right)\right]}{4\left[8m^5-12Q^2m^3+4Q^4m+\sqrt{m^2-Q^2}\left(8m^4-8Q^2m^2+Q^4\right)\right]}\theta^2 \quad (5.11)$$

If we consider the Q = 0 case, then we obtain the corresponding quantities for Schwarzschild black holes.

6. Concluding remarks

Using the Seiberg-Witten map we determined the noncommutativity corrections to the Schwarzschild solution up to the second order in the parameters $\theta^{\mu\nu}$. Following the Ref. [7], we reviewed first the de-Sitter gauge theory of gravitation over a spherical symmetric commutative Minkowski space-time. Then, a deformation of the gravitational field has been constructed along the Ref. [8] by gauging the noncommutative de-Sitter SO(4,1) group and using Seiberg-Witten map. The corresponding space-time is also of Minkowski type but endowed now with spherical noncommutative coordinates. We determined the deformed gauge fields up to the second order in the noncommutativity parameters $\theta^{\mu\nu}$. The deformed gravitational gauge potentials (tetrad fields) $\hat{e}^a_\mu(x,\theta)$ have been obtained by contraction of the noncommutative gauge group SO(4,1) to the Poincaré (inhomogeneous Lorentz) group ISO(3,1). As an application, we calculated these potentials for the case of the Schwarzschild solution and defined the corresponding deformed metric $\hat{g}_{\mu\nu}(x,\theta)$. The corrections appear only in the second order of the expansion in θ, i.e. there are no first order correction terms. For the calculations we used an analytical program conceived for the GRTensor II package of the Maple platform.

We have considered also the red shift test in the noncommutative theory and determined the value of the relative displacement $\frac{\Delta\lambda}{\lambda}$ for the case of Sun. The result shows that the correction is too small to have observable effects.

Having found the deformed solutions for a noncommutativity theory of gravy we have been breaking new ground towards approaching the black hole physics on non-commutative space-time [29].

The corrections for the event horizon radius, Hawking temperature and the entropy of the black hole have been evaluated. It is important to emphasize that in the case of Reissner-Nordström black hole we considered the commutative (non-deformed) electromagnetic field as a first step to obtain the solution. It remains, as an open question, to give a method to introduce into the model a deformed electromagnetic field.

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