Classical and Quantum Models for the Gauge Fields
- Stage 3 -

Objective 1. Construction of the gauge invariant integral action

1.1. The calculation of the integral action for the scalar field in the presence of gauge fields

The gravitational field can be described like other fields (electromagnetic for example) by using the gauge theory. For such a purpose one can use as gauge group (of local symmetry) the Poincaré group $ISO(3,1)$, the de-Sitter group $SO(4,1)$ or anti-de-Sitter $SO(3,2)$, the group of affine transformations $A(4,R)$, etc. The main problem appearing in such a theory is the quantization of the gravitational field. Any quantum theory, in particular those of the gravitational field, must be renormalizable. This property – renormalizability – is one of the most important problems of the quantum theory of gravitational field which is expected to be solved yet.

One possible way for obtaining a quantum theory of the gravitational field is to consider the gravitational gauge group $[ISO(3,1), SO(3,2), etc.]$ as a purely inner symmetry [10]. This means that the space-time coordinates do not change under the gauge group transformations, but the transformation formulas of the gauge and matter fields are correspondingly modified.

In our project [stage 3/2009] we used as group of local symmetry a deformation of the de-Sitter group $SO(4,1)$ determined by a real parameter λ. This allows us to introduce a cosmological constant Λ into the model whose value is precisely determined by the parameter λ [1]. In the limit $\lambda \to 0$ (a process which is named group contraction) the cosmological constant Λ vanishes and the de-Sitter group contracts to the Poincaré group $ISO(3,1)$. This means that a theory of the gravitational field based on the Poincaré group $ISO(3,1)$ is not adequate to construct cosmological models [10]. Thus, in our works we used the de-Sitter group (a deformation) in order to describe the gravitation by a model including the cosmological constant.

The infinitesimal generators of the de-Sitter group $SO(4,1)$ are $J_{ab} = -J_{ba}$, $a,b = 0,1,2,3,4,5$; in what follows we denote them by $\Pi_\alpha \equiv J_{\alpha 5}$ and $M_{\alpha\beta} \equiv J_{\alpha\beta} = L_{\alpha\beta} + \frac{1}{2}\Sigma_{\alpha\beta}$, $\alpha,\beta = 0,1,2,3$, where $L_{\alpha\beta}$ are interpreted as the angular momentum operators and $\Sigma_{\alpha\beta}$ as the spin operators in a m-dimensional representation spanned by the matter field we are considering (denoted below by φ_j, $j = 1,2,...,m$). In the limit $\lambda \to 0$ the generators Π_α pass in space-time translations generators P_α corresponding to the momentum operators and $M_{\alpha\beta}$ do not change; they generate the Lorentz transformations and are interpreted as the total angular momentum operators.

Then, we consider a multiplet of matter fields φ_j, $j = 1,2,...,m$, whose dynamics is described by the Lagrangian $L_M(\varphi_j, \partial_\alpha \varphi_j)$ and construct the integral of action, supposed to be global invariant under the group $SO(4,1)$, in the usual form

$$S_M = \int d^4x\, L_M(\varphi_j, \partial_\alpha \varphi_j), \tag{1}$$

where $x = (x^\mu)$, $\mu = 0,1,2,3$ denotes the coordinates on the Minkowski space-time. Let us now suppose that the action S_M is also invariant under the gauge (local) transformations of the de-Sitter group $SO(4,1)$. In contrast with the usual models considered by other authors we will write the de-Sitter transformations (which are space-time transformations) as purely inner transformations [1], i.e.:

$$x^\alpha \to x'^\alpha = x^\alpha, \tag{2a}$$

$$\varphi_j(x) \to \varphi'_j(x) = ((1+\Theta)\varphi_j)(x), \tag{2b}$$

where

$$\begin{aligned}\Theta &= -[\varepsilon^\gamma(x) + \lambda^2 \varepsilon^\beta(x) t^\gamma_\beta + \omega^{\gamma\delta}(x) x_\delta]\partial_\gamma - \frac{i}{4}\omega^{\gamma\delta}(x)\Sigma_{\gamma\delta} + i\lambda\varepsilon^\gamma(x)\Sigma_{\gamma 5} \\ &= i\varepsilon^\gamma(x)\Pi_\gamma - \frac{i}{2}\omega^{\gamma\delta}(x) M_{\gamma\delta}\end{aligned} \tag{3}$$

Here, the quantities t^{α}_{β} depend only on the space-time coordinates x^{μ}, $\mu = 0,1,2,3$ and they have the expression

$$t^{\gamma}_{\beta} = 2\eta_{\beta\delta}x^{\delta}x^{\gamma} - \sigma^2\delta^{\gamma}_{\beta}; \quad \sigma^2 = \eta_{\mu\nu}x^{\mu}x^{\nu}, \tag{4}$$

$\eta_{\mu\nu} = diag(-1,1,1,1)$ being the Minkowski metric. In the last expression of Θ from equation (3), the quantities $\varepsilon^{\gamma}(x)$ and $\omega^{\gamma\delta}(x) = -\omega^{\delta\gamma}(x)$ denotes the infinitesimal parameters (which depends on coordinates) of the de-Sitter gauge group $SO(4,1)$.

As usual, we introduce the gravitational gauge fields (potentials) $B^{\gamma}_{\alpha}(x)$ and $B^{\gamma\delta}_{\alpha}(x) = -B^{\delta\gamma}_{\alpha}(x)$ which correspond respectively to the generators Π_{γ} and $M_{\gamma\delta}$. Then, we define the gauge covariant derivative

$$\nabla_{\alpha} = \partial_{\alpha} + B_{\alpha}, \quad B_{\alpha} = -iB^{\gamma}_{\alpha}\Pi_{\gamma} + \frac{i}{2}B^{\gamma\delta}_{\alpha}M_{\gamma\delta}. \tag{5}$$

Equivalent, this derivative can be written under the form [1, 7, 9]

$$\nabla_{\alpha} = e^{\gamma}_{\alpha}\partial_{\gamma} + \frac{i}{4}B^{\gamma\delta}_{\alpha}\Sigma_{\gamma\delta} - i\lambda B^{\gamma}_{\alpha}\Sigma_{\gamma 5}, \tag{6}$$

where

$$e^{\gamma}_{\alpha} = \delta^{\gamma}_{\alpha} + B^{\gamma}_{\beta} + \lambda^2 B^{\beta}_{\alpha}t^{\gamma}_{\beta} + B^{\gamma\delta}_{\alpha}x_{\delta}. \tag{7}$$

The transformation laws of the gauge fields B^{γ}_{α}, $B^{\gamma\delta}_{\alpha}$ and implicitly e^{γ}_{α} under the de-Sitter gauge group $SO(4,1)$ are obtained in our work [1].

In order that the integral of action (1) to be also invariant under the de-Sitter gauge (local) group de-Sitter we must change in the Lagrangian $L_M(\varphi_j, \partial_{\alpha}\varphi_j)$, initial supposed global invariant, the ordinary differential derivative ∂_{α} by the gauge covariant derivative ∇_{α} and introduce the factor $e^{-1} = \det\left(e^{-1\gamma}_{\ \ \varepsilon}\right)$ under the integral, where $e^{-1\gamma}_{\ \ \varepsilon}$ denotes the inverse matrix of $\varepsilon^{\gamma}_{\alpha}$, i.e. $e^{\varepsilon}_{\alpha}e^{-1\gamma}_{\ \ \varepsilon} = \delta^{\gamma}_{\alpha}$. Therefore, the minimal extended action and invariant under the de-Sitter gauge group $SO(4,1)$ has the expression

$$S_M(\varphi_j; e) = \int d^4x\, e^{-1}L_M(\varphi_j(x), \nabla_{\alpha}\varphi_j(x)). \tag{8}$$

In particular, if we consider a scalar real matter field $\varphi(x)$ with mass *m*, then the gauge invariant action under the de-Sitter group (considered as a purely inner symmetry group), has the form

$$S_M(\varphi; e) = \int d^4x\, e^{-1}\left(d_{\alpha}\varphi d^{a}\varphi - \frac{1}{2}m^2\varphi^2\right), \tag{9}$$

where $d_{\alpha} = e^{\gamma}_{\alpha}\partial_{\gamma}$. Here, $\nabla_{\alpha}\varphi$ reduces to $d_{\alpha}\varphi$ because the scalar field belongs to the trivial unit representation.

1.2. The calculation of the integral action for the spinorial field in the presence of the gauge fields

The concept of purely inner $SO(4,1)$ symmetry together with the gauge principle allows us to describe as previously the minimal coupling of the spinorial and gauge gravitational fields. In conformity with the concept of purely inner symmetry $SO(4,1)$, the gauge fields do not interfere with the space-time structure a priori fixed by our convention (of purely inner symmetry) and the geometry of this space remains separated from the physics described by the $SO(4,1)$ gauge fields.

Let us consider now a spinorial field $\psi(x)$ having the mass *m*. The global invariant action under the $SO(4,1)$ group is given by the expression

$$S_M = \int d^4x\left[\frac{i}{2}\overline{\psi}\gamma^{\alpha}(\partial_{\alpha}\psi) - \frac{i}{2}(\partial_{\alpha}\overline{\psi})\gamma^{\alpha}\psi - m\overline{\psi}\psi\right]. \tag{10}$$

Here, γ^{α} are the Dirac matrices satisfying the usual Clifford algebra $\{\gamma^{\alpha}, \gamma^{\beta}\} = 2\eta^{\alpha\beta}$ and the spin operators have the expressions $\Sigma_{\alpha\beta} = \frac{i}{2}[\gamma_{\alpha}, \gamma_{\beta}]$.

Due to our previous hypothesis of minimal coupling, we can write the gauge invariant action of the spinorial field $\psi(x)$ under the form

$$S_M(\psi,\overline{\psi};e,B)=\int d^4x e^{-1}\left[\frac{i}{2}\overline{\psi}\gamma^\alpha(\nabla_\alpha\psi)-\frac{i}{2}(\nabla_\alpha\overline{\psi})\gamma^\alpha\psi-m\overline{\psi}\psi\right]. \quad (11)$$

Because $\psi(x)$ is a spinorial field, the action (11) includes now the gravitational gauge fields B_α^γ and $B_\alpha^{\gamma\delta}$ through the gauge covariant derivative ∇_α.

1.3. The calculation of the action integral for the massive vector field in the presence of the gauge fields

We consider now the case of a vector field $A_\mu(x)$, $\mu=0,1,2,3$ with non-null mass *m*. The integral of action which is global invariant under the de-Sitter group $SO(4,1)$ has the expression

$$S_M=\int d^4x\left[-\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}+\frac{1}{2}m^2A_\alpha A^\alpha\right],$$

where $F_{\alpha\beta}=\partial_\alpha A-\partial_\beta A_\alpha$ is the strength field tensor associated to $A_\alpha(x)$. In particular, we can consider the gauge fields $A_\alpha(x)=A_\alpha^k(x)T_k$ with values in the Lie algebra, associated to an internal symmetry group, for example $SU(n)$, having the infinitesimal generators $T_k, k=1,2,...,N$. The spin operators have in this case the expressions $(\Sigma_{\alpha\beta})^{\gamma\delta}=2i(\eta_\alpha^\gamma\eta_\beta^\delta-\eta_\alpha^\delta\eta_\beta^\gamma)$ which correspond to the vector representation of the de-Sitter group. The integral of action gauge invariant under the de-Sitter symmetry group corresponding to the minimal coupling is

$$S_M(A;e,B)=\int d^4x\left[-\frac{1}{4}\widetilde{F}_{\alpha\beta}\widetilde{F}^{\alpha\beta}+\frac{1}{2}m^2A_\alpha A^\alpha\right], \quad (12)$$

where $\widetilde{F}_{\alpha\beta}=\nabla_\alpha A_\beta-\nabla_\beta A_\alpha$. The gauge covariant derivative of the vector field $A_\alpha(x)$ has the expression $\nabla_\alpha A_\beta=d_\alpha A_\beta-B_{\alpha\beta}^\gamma A_\gamma+\lambda B_\alpha^\gamma A_\gamma$. If $A_\alpha(x)$ is a $U(1)$ gauge field, then its transformation law under the group $U(1)$ is $A_\alpha\to A_\alpha+\nabla_\alpha\theta(x)$, where $\theta(x)$ is the parameter (which depends on coordinates) of $U(1)$. It is important to observe that in general the tensor $\widetilde{F}_{\alpha\beta}$ is not invariant under $U(1)$.

In the case when the gauge theory is defined on a non-commutative (NC) [5, 6] we used the covariant star-product $*$ [4] between fields. The integral of action for the gauge fields $A_\alpha(x)$ has then the expression

$$S_{NC}=-\frac{1}{2g^2}\int d^4x\hat{G}^{\alpha\beta}*\hat{F}_{\beta\gamma}*G^{\gamma\delta}*\hat{F}_{\delta\alpha}, \quad (13)$$

where $\hat{G}^{\alpha\beta}$ denotes the gauge covariant metric of the NC space-time, $\hat{F}_{\alpha\beta}$ is the strength tensor of the gauge fields $A_\alpha(x)$ and *g* is the gauge coupling constant. In the case of a matter scalar field $\varphi(x)$, a possible form of the integral of action which corresponds to a renormalizable model in all orders of a perturbative theory is [15]

$$S_\varphi=\int d^4p\left(\frac{1}{2}p_\alpha\varphi\, p^\alpha\varphi+\frac{1}{2}\left(m^2\varphi\varphi+a\frac{1}{\theta^2p^2}\right)\varphi\varphi+\frac{f}{4!}\varphi*\varphi*\varphi*\varphi\right).$$

Applications to the case of gauge fields with spherical symmetry are given in our papers [2, 3, 4, 8].

Objective 2. The calculation of the matter partition function in the presence of the gauge fields

2.1. Obtaining the one-loop partition function for the scalar field in the presence of the gauge fields

In our works we determined the one-loop partition function $Z_\varphi[e]$ and studied the behaviour of this function under a scale transformation (re-scaling). The contribution of the scalar field $\varphi(x)$ to the partition function is given by the expression [11]

$$Z_\varphi[e]=\int D\varphi e^{iS_M(\varphi;e)}, \quad (14)$$

where $S_M(\varphi;e)$ is the gauge invariant action under the de-Sitter group of the scalar field $\varphi(x)$ given in (9). After a partial integration the formula (9) can be written in the form

$$S_M(\varphi;e)=\frac{1}{2}(\varphi,M_\varphi(e)\varphi), \quad (15)$$

where $M_{\varphi}(e) = -\nabla_{\alpha}\nabla^{\alpha} - m^2$ is the field hyperbolic operator. In (15) we denoted by (χ, φ) the scalar product on the space of real scalar functions: $(\chi, \varphi)\int d^4x e^{-1}\chi\varphi$. After a Gaussian integration the expression (15) can be written in the standard form

$$Z_{\varphi}[e] = \exp\left[\frac{1}{2}\ln\det M_{\varphi}(e)\right]. \tag{16}$$

On the other hand, the functional determinant [12] of an operator M can be expressed through the generalized zeta-function ζ as follows:

$$\ln\det M = -\lim_{u\to 0}\frac{d}{du}\zeta(u;\mu;M), \tag{17}$$

where, by definition, $\zeta(u;\mu;M) = \mu^{2u} trM^{-u}$ and μ is a scale factor. Considering our operator $M_{\varphi(e)}$, we can write (introducing the scale factor μ)

$$Z_{\varphi}[\mu;e] = \exp[\zeta'(0;\mu;M_{\varphi}(e))]. \tag{18}$$

Making a scale transformation $\mu \to \tilde{\mu} = \lambda\mu$, the partition function Z_{φ} becomes [7, 9]

$$Z_{\varphi}[\tilde{\mu};e] = Z_{\varphi}[\mu;e]\exp[\ln\lambda\,\zeta(0;\mu;M_{\varphi}(e))]. \tag{19}$$

On the other hand, the zeta-function $\zeta(u;\mu;M)$ can be expressed as Mellin transformation of the heat kernel [13]

$$\zeta(u;\mu;M) = \frac{i\mu^{2u}}{\Gamma(u)}\int_0^{\infty} ds(is)^{u-1} Tr\exp(isM). \tag{20}$$

Using this expression we obtain the following transformation property of the zeta-function ζ to the re-scaling $\mu \to \tilde{\mu} = \lambda\mu$ [9]

$$\zeta'(0;\tilde{\mu};M) = \zeta'(0;\mu;M) + 2\ln\lambda\,\zeta(0;\mu;M). \tag{21}$$

This result shows that the transformation of the functional determinant [see the definition (17)] under re-scaling is fully determined by $\zeta(0;\mu;M)$. Therefore, in order to analyse the renormalizability property of our de-Sitter gauge theory it is necessary to calculate first the function $\zeta(0;\mu;M)$ using for M one of the above operators: $M_{\varphi}(e)$ - for the scalar field [Eq. (15)], $M_{\psi}(e,B)$ - for the spinorial field [Eq. (23)], or $M_F(e,B)$ - for the case of the vector field [Eq. (29)]. The results will be presented below into the **Objective 3. The study of renormalizability properties of the formulated theory**.

2.2. Obtaining the one-loop partition function for the spinorial field in presence of the gauge fields

In the case of a spinorial field $\psi(x)$ the partition function is given by the Grassmann integral functional [7]

$$Z_{\psi}[e,B] = \int D\overline{\psi}\, D\psi \exp[iS_M(\psi,\overline{\psi};e,B)], \tag{22}$$

where $S_M(\psi,\overline{\psi};e,B)$ is the gauge invariant action in (11). As previous, we can write

$$Z_{\psi}[e,B] = \exp\left[\frac{1}{2}\ln\det M_{\psi}(e,B)\right], \tag{23}$$

where the hyperbolic operator of field fluctuations $M_{\psi}(e,B)$ has the form [9]

$$M_{\psi}(e,B) = -D_{\alpha}D^{\alpha} + \frac{i}{2}F_{\alpha\beta}\Sigma^{\alpha\beta} - m^2. \tag{24}$$

Here, $D_{\alpha} = \nabla_{\alpha} + B_{\alpha}$ and $F_{\alpha\beta}$ is the 2-forma of curvature of the $SO(4,1)$ gauge fields $B_{\alpha}^{\gamma}(x)$ and $B_{\alpha}^{\gamma\delta}(x) = -B_{\alpha}^{\delta\gamma}(x)$ [9]. Using the definition (17), we obtain from (23) the following expression of the partition function of a spinorial field with the scale factor μ

$$Z_{\psi}[\mu;e,B] = \exp\left[-\frac{1}{2}\zeta'(0;\mu;M_{\psi}(e,B))\right], \tag{25}$$

Using the property (21) we deduce the transformation formula of the partition function $Z_\psi[\mu;e,B]$ under the re-scaling $\mu \to \tilde{\mu} = \lambda\mu$

$$Z_\psi[\tilde{\mu};e,B] = Z_\psi[\mu;e,B]\exp[-\ln\lambda\,\zeta(0;\mu;M_\psi(e,B))]. \tag{26}$$

Therefore, in order to analyse the renormalizability property of our de-Sitter gauge theory, we must determine the zeta-function $\zeta(0;\mu;M_\psi(e,B))$. The results will be presented below in **Objective 3** where we study the renormalizability using the one-loop partition function for the spinorial field.

2.3. Obtaining the one-loop partition function for massive vector field in presence of gauge fields

The contribution of a massive vector field A_α to the partition function has the expression

$$Z_A[e,B] = \int DA_\alpha \exp[iS_M(A;e,B)], \tag{27}$$

where $S_M(A;e,B)$ is the gauge invariant action in presence of gauge fields [Eq. (13)]. In order to calculate this functional we use the Fadeev-Popov method [12]. Namely, we chose a gauge determined by a gauge condition $F[A_A^\theta] = G(x)$ and insert the unit $1 = \int D\theta\, DA_\alpha\, \delta(F[A_A^\theta] - G(x))\det M_F(A)$ into the expression (27). Then, we obtain

$$Z_A[e,B] = \int D\theta\, DA_\alpha\, \delta(F[A_\alpha^\theta] - G(x))\det M_F(A)\exp[iS_M(A;e,B)]. \tag{28}$$

Because the Fadeev-Popov determinant $M_F(A)$ is gauge invariant, we can make the transformation $A_\alpha \to A_\alpha + \nabla_\alpha\theta$ in Eq. (28) without affecting the result. In particular, if we chose the gauge condition in the form $F[A_\alpha^\theta] = -\nabla_\alpha A^\alpha$, then the Fadeev-Popov operator $M_F(e,B) = -\nabla_\alpha\nabla^\alpha$, the integral (28) becomes of Gaussian type and can be calculated resulting

$$Z_A[e,B] = \det M_F(e,B)\exp\left[-\frac{1}{2}\ln\det M_F(e,B)\right]. \tag{29}$$

Like in the previous two cases, we can prove that the transformation of this functional under a re-scaling $\mu \to \tilde{\mu} = \lambda\mu$ is fully determined by the zeta-function $\zeta(0;\mu;M_F(e,B))$. We present tits calculation below in **Objective 3**.

Objective 3. The study of properties for the developed theory

3.1. Analysis of the renormalizability theory

The calculation of the effective quantum action for different types of fields which can be regularized and renormalized at the one-loop level, we can use the generalized ζ -function [1, 9]. In many cases (for example in the calculation of the effective action) this method naturally leads to the vanishing of the divergences maintaining physical terms in the result. In particular, if we consider the case of the scalar field $\varphi(x)$, whose hyperbolic operator of fluctuations is $M_\varphi(e)$, then the effective action is given by the derivative of the ζ -function calculated for $u = 0$ [14]

$$S_{eff}[\varphi,e] = \frac{1}{2}\frac{d}{du}\zeta(u;\mu;M_\varphi(e))_{|u=0}. \tag{30}$$

It is important to remark that there is an important relationship between the presence of poles for ζ -function and the coefficients $c_k(x), k = 0,1,2,.....$ of the heat kernel $K(u;x,y)$. Using these properties, we present below our results for the coefficients c_1, c_2 $(c_0 = 1)$ and determine the effective minimal action for the scalar, , spinorial and massive vectorial fields which is compatible with renormalizability requires up to the one-loop level.

3.2. Obtaining the anomaly terms

In order to obtain the function $\zeta(0;\mu;M)$, which fully determines the partition function, we use the expression (20) of the ζ -function. Then, introducing the result into the expression of functional integral and taking the limit $u \to 0$ we obtain the singular terms which are the anomalies of the considered model. Renormalizability of any theory which includes dynamical gauge fields requires that these anomalies, which are polynomials of gauge fields (in our case e_α^γ and $B_\alpha^{\gamma\delta}$) and their derivatives, can be absorbed in classical action of the gauge fields. Therefore, in order to determine explicitly the dynamics of the gauge fields in accord with the renormalization requirements, we must determine the corresponding ζ-functions.

We write the fluctuation operator in the general form $M = D_\alpha D^\alpha + E$, $D_\alpha = \nabla_\alpha + N_\alpha$ where E and N_α are the operators whose expressions depend on the type of the considered fields (scalar, spinorial, vectorial). We remember that the strength tensor $F_{\alpha\beta}$, associated to our SO(4,1) gauge fields has the components: $F^\gamma_{\alpha\beta} \equiv T^\gamma_{\alpha\beta}; F^{\gamma\delta}_{\alpha\beta} \equiv R^{\gamma\delta}_{\alpha\beta}$ (these notations show the relation with the geometrical models defined on Riemann-Cartan space-times with torsion $T^\gamma_{\alpha\beta}$ and curvature $R^{\gamma\delta}_{\alpha\beta}$). In our works we considered only the case $T^\gamma_{\alpha\beta} = 0$, i.e. the torsion vanishes.

The heat kernel $K(u;x,y)$ satisfies the equation

$$\left(\frac{\partial}{\partial u} + M_x\right)K(u;x,y) = 0\,, \tag{31}$$

where M_x denotes the derivative of the operator M with respect of the x variable. In the limits $y \to x$ and $u \to 0$ we can use the asymptotic expansion

$$K(u;x,y) \approx \frac{i}{(4\pi u)^{d/2}} e^{-\frac{r^2(x,y)}{4u}} \sum_{k=0}^{\infty} u^k c_k(x,y). \tag{32}$$

Here, d denotes the dimension of the space-time which in our case will be d = 4. The expressions of the $r^2(x,y)$ function and the coefficients $c_k(x,y)$ in the limit $y \to x$ are calculated by imposing the condition that (32) verifies the heat kernel equation (31). By a direct but laborious calculus, we obtained the expressions

$$\begin{aligned} c_1(x) &= -\frac{1}{6}R - E, \quad c_2(x) = -\frac{1}{30}\nabla_\alpha \nabla^\alpha R^{\alpha\beta}{}_{\alpha\beta} + \frac{1}{72}R_{\alpha\beta}{}^{\alpha\beta} R_{\gamma\delta}{}^{\gamma\delta} + \frac{1}{180}R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} \\ &- \frac{1}{180}R_{\alpha\gamma}{}^{\alpha}{}_{\delta} R_\beta{}^{\gamma\beta\delta} + \frac{1}{12}\widetilde{F}_{\alpha\beta}\widetilde{F}^{\alpha\beta} + \frac{1}{6}R^{\alpha\beta}{}_{\alpha\beta}E - \frac{1}{6}\left[D_\alpha, \left[D^\alpha, E\right]\right] + \frac{1}{2}E^2 \end{aligned} \tag{33}$$

where $\widetilde{F}_{\alpha\beta} = \nabla_\alpha A_\beta - \nabla_\beta A_\alpha + \left[A_\alpha, A_\beta\right]$.

Because the anomalies come from the region corresponding to small values of u ($u \to 0$), we can use the asymptotic formula (32) for the heat kernel. Then, by partial integrating with respect to u of (20) and maintaining only the contribution $k = \frac{1}{2}$ in the sum over k, we obtain

$$\zeta(0;\mu;M) = \frac{i}{(4\pi)^{d/2}} \int d^d e^{-1} Tr c_{\frac{d}{2}}(x). \tag{34}$$

3.3. Applications of the obtained results to the case of the scalar, spinorial and vectorial fields

In the case of the scalar field the operator of fluctuations $M_\varphi(e)$ can be obtained from above expression of M choosing: $E = -m^2$ and $N_\alpha = 0$. Then, using Eq. (34) we calculate $\zeta(0;\mu;M_\varphi(e))$ and introducing the obtained expression in Eq. (18) we obtain the anomaly term for the scalar field. For the spinorial and vectorial fields the method of calculus is analogous, but the operators E and N_α have specific forms. For example, for spinorial field we have $N_\alpha = \frac{i}{4}\Sigma_{\beta\gamma}\left(B^{\beta\gamma}_\alpha + T_\alpha{}^{\beta\gamma}\right)$ and this expression simplifies if $T_\alpha{}^{\beta\gamma} = 0$ (vanishing torsion). The results are contained in our works [1, 7, 9].

Using these results we can construct the minimal action for the gauge fields. Like in any classical gauge field dynamics, consistent with the renormalizability, there are present anomalous terms. In our model, considering $T_\alpha{}^{\beta\gamma} = 0$, the minimal classical action for the gauge fields e^γ_α and $B^{\gamma\delta}_\alpha$ associated to the $SO(4,1)$ group must contain the following terms, if we neglect the total divergences [7]

$$S_{gauge}(e,B) = \int d^4x\left[-\frac{1}{16\pi G}(R - 2\Lambda) + aR^2 + bR_{\alpha\gamma}{}^{\alpha}{}_{\delta}R_\beta{}^{\gamma\beta\delta} + cR_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}\right]. \tag{35}$$

Here, G is the gravitational constant and a,b,c are coupling constants. We can see that by using the de-Sitter gauge group $SO(4,1)$, the cosmologic constant automatically enter into the expression of action; it has the value

$\Lambda = -12\lambda^2$ [1, 9]. It is important to remark that the action integral S_{gauge} from Eq. (35) is on one hand invariant with respect to the gauge (local) de-Sitter transformations and, on the other hand, it is invariant with respect to global Poincaré transformations on the Minkowski space-time.

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